A Loss Reserving Method for Incomplete Claim Data

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June 11, 2008

Abstract

A stochastic model of an additive loss reserving method based on the assumption that the claim reserves are good measures for the remaining exposure is presented. This model combines a projection of payments and a projection of corresponding reported amounts such that both leads to the same ultimate. In addition, the presented method even works for some kind of incomplete triangles.

We will state estimators for the total necessary reserves and estimators for the corresponding standard error. Moreover, we will discuss an example based on motor liability data, where we distinguish between property damage and bodily injury claims and only have proper data to do so for the last six accounting years.

KEYWORDS: STOCHASTIC RESERVING, LOSS DEVELOPMENT, MEAN SQUARED ERROR OF PREDICTION.

1 Motivation

We want to describe a method which is able to deal with some kind of incomplete triangles of payments and reported amounts. As an example you may keep in mind the following situation. Assume your company covers motor liability. Some years ago one decided to distinguish between bodily injury and property damage claims. All new claims, all still open claims and all reopened claims have been migrated to the new claims handling philosophy. This implies that for old years only some claims can be identified as bodily injury or property damage claims and therefore one cannot apply the Chain Ladder method on the separated triangles of bodily injury and property damage claims.

For a more precise definition let $S_{i,k}$ and $T_{i,k}$ denote the incremental payments and incremental reported amounts of accident year i and development year k, for $1 \leq i, k \leq n$. Here, n denotes the number of observed years. By incomplete data we mean that $S_{i,k}$ and $T_{i,k}$ are missing, for $i + k \leq Y$ for some accounting year $Y < n$. Clearly, in this situation the Chain Ladder method doesn't work properly. But, if a good measure of the underlying risk for each accident year can be obtained, the Complementary Loss Ratio method, see [3], could be applied to both the incremental triangle of payments and the incremental triangle of reported amounts.

We will extend the idea of the Complementary Loss Ratio method using the case reserves (outstanding) at the end of the previous development year as risk measure. In other words, payments and adjustments to the reported amount during the year are assumed to be proportional to the opening reserves, which are the case reserves at the end of the previous year. In order to do this we need all case reserves $R_{i,Y-i}$, $i < Y$. We will present unbiased estimators for the total necessary reserves. Moreover, we will use the ideas of Mack [2] to present an estimator for the corresponding conditional mean squared error.

2 Definition of the model

We want to extend the idea of incomplete triangles in the way, that we introduce non-negative weights $w_{i,k}$, $i + k \leq n$, which measures our confidence in the observed developing during the year k of accident year i. The situation of the incomplete triangles described in the previous section is the special case of weights $w_{i,k} = 0$, for $i + k \leq Y$, and $w_{i,k} = 1$, for $Y < i + k \leq n$. We will use the following basic notations:

- $S_{i,k}$ the payments during development year k of all claims of accident year i.
- $T_{i,k}$ the adjustments to the reported amount during development year k of all claims of accident year i.

$R_{i,k}$ the outstanding (case reserves) at the end of development year k of all claims of accident year i.

We consider $S_{i,k}$, $T_{i,k}$ and $R_{i,k}$ as random variables for which we have observations for $i+k \leq n+1$. This variables are linked via

$$R_{i,k+1} = R_{i,k} - S_{i,k+1} + T_{i,k+1}, \qquad \text{for } 1 \leq i \leq n \text{ and } 0 \leq k < n,$$

where we take $R_{i,0} \equiv 0$.

As usual we assume the accident years, i.e.

$$\{S_{1,k},\, T_{1,k},\, R_{1,k} \colon 1 \leq k \leq n\}, \ldots, \{S_{n,k},\, T_{n,k},\, R_{n,k} \colon 1 \leq k \leq n\}, \tag{2.1}$$

are independent.

The minimum and maximum of two real numbers we denote by

$$a \vee b := \max(a,b) \qquad \text{and} \qquad a \wedge b := \min(a,b).$$

Moreover, we define the information $\mathcal{D}_k$ of the run-off triangles up to development year $1 \leq k \leq n$ by

$$\mathcal{D}_k := \sigma\left(\bigcup_{i=1}^{n} \mathcal{B}_{i,k\wedge(n+1-i)}\right) \tag{2.2}$$

with

$$\mathcal{B}_{i,k} := \sigma\left(\{S_{i,l},\, T_{i,l} \colon 1 \leq l \leq k\}\right), \tag{2.3}$$

the run-off information of accident year i up to development year k.

The idea of case reserves being a good risk measure for incremental payments and reported amounts can be formalised as follows:

Assumption 2.1 *For $1 \leq i \leq n$ and $1 \leq k < n$ we assume that*

$$E\left[S_{i,k+1} | \mathcal{B}_{i,k}\right] = \alpha_k R_{i,k}\,, \tag{2.4}$$

$$E\left[T_{i,k+1} | \mathcal{B}_{i,k}\right] = \beta_k R_{i,k}\,, \tag{2.5}$$

$$\mathrm{Cov}\left[\begin{pmatrix} S_{i,k+1} \\ T_{i,k+1} \end{pmatrix}, \begin{pmatrix} S_{i,k+1} \\ T_{i,k+1} \end{pmatrix} \middle| \mathcal{B}_{i,k}\right] = \begin{pmatrix} \sigma_k^2 & \gamma_k \\ \gamma_k & \tau_k^2 \end{pmatrix} R_{i,k} =: \Sigma_k^2 R_{i,k} \tag{2.6}$$

for some constants α_k and β_k and some positive definite, symmetric matrices Σ_k.

Assumption (2.6) ensures that our weighted estimator for the necessary reserves will be of minimal variance under all linear estimators. Moreover, (2.6) also implies that all reserves have to be non-negative and together with (2.4) and (2.5) we even get that all reserves have to be positive up to the time where no development is assumed to be taken place.

Standard calculations using conditional expectations gives us.

Corollary 2.2 *Assumption 2.1 implies the standard Chain Ladder assumptions for the reserve, i.e.*

$$E[R_{i,k+1}|\mathcal{B}_{i,k}] = (1-\alpha_k+\beta_k)R_{i,k} := f_k R_{i,k}\,, \tag{2.7}$$

$$\mathrm{Var}[R_{i,k+1}|\mathcal{B}_{i,k}] = (\sigma_k^2 - 2\gamma_k + \tau_k^2)R_{i,k}\,. \tag{2.8}$$

Remark 2.3

1) *Hence Corollary 2.2 immediately implies* $1-\alpha_k+\beta_k \geq 0$*, for all* $k \geq 1$.
2) *Looking at cumulative data* $C_{i,k} := \sum_{j=1}^{k} S_{i,j}$ *and* $D_{i,k} := \sum_{j=1}^{k} T_{i,j}$ *the assumptions* (2.4) *and* (2.5) *imply that*

$$\begin{aligned} E[C_{i,k+1}|\mathcal{B}_{i,k}] &= (1-\alpha_k)C_{i,k} + \alpha_k D_{i,k}\,,\\ E[D_{i,k+1}|\mathcal{B}_{i,k}] &= (1+\beta_k)D_{i,k} - \beta_k C_{i,k}\,. \end{aligned}$$

3 Estimator of the necessary reserves

The main aim of this section it to define weighted estimators for the parameter α_k and β_k and for the unknown values $S_{i,k}$, $T_{i,k}$ and $R_{i,k}$, for $i+k > n+1$. Moreover, we will derive some useful conclusions.

Proposition 3.1 *Under the assumptions stated in Section 2*

$$\widehat{\alpha}_k := \frac{\sum_{i=1}^{n-k} w_{i,k} S_{i,k+1}}{\sum_{i=1}^{n-k} w_{i,k} R_{i,k}} \qquad \text{and} \tag{3.1}$$

$$\widehat{\beta}_k := \frac{\sum_{i=1}^{n-k} w_{i,k} T_{i,k+1}}{\sum_{i=1}^{n-k} w_{i,k} R_{i,k}} \tag{3.2}$$

are conditionally unbiased estimators for α_k *and* β_k*, respectively, given* $\mathcal{D}_k$.

Moreover, $\widehat{f}_k := (1-\widehat{\alpha}_k+\widehat{\beta}_k)$ *is the Chain-Ladder estimator for* f_k*, that means*

$$\widehat{f}_k = \frac{\sum_{i=1}^{n-k} w_{i,k} R_{i,k+1}}{\sum_{i=1}^{n-k} w_{i,k} R_{i,k}}. \tag{3.3}$$

The estimators $\widehat{\alpha}_k$, $\widehat{\beta}_k$ *and* $\widehat{f}_k$ *are correlated via*

$$\mathrm{Cov}\left[\begin{pmatrix}\widehat{\alpha}_k\\ \widehat{\beta}_k\\ \widehat{f}_k\end{pmatrix}, \begin{pmatrix}\widehat{\alpha}_l\\ \widehat{\beta}_l\\ \widehat{f}_l\end{pmatrix} \middle| \mathcal{D}_{l\wedge k}\right] = 0, \qquad \text{for } k \neq l, \tag{3.4}$$

and

$$\mathrm{Cov}\left[\begin{pmatrix}\widehat{\alpha}_k\\ \widehat{\beta}_k\\ \widehat{f}_k\end{pmatrix},\begin{pmatrix}\widehat{\alpha}_k\\ \widehat{\beta}_k\\ \widehat{f}_k\end{pmatrix}\middle|\mathcal{D}_k\right]=\begin{pmatrix}\sigma_k^2 & \gamma_k & \gamma_k-\sigma_k^2\\ \gamma_k & \tau_k^2 & \tau_k^2-\gamma_k\\ \gamma_k-\sigma_k^2 & \tau_k^2-\gamma_k & \sigma_k^2-2\gamma_k+\tau_k^2\end{pmatrix}\frac{\sum_{j=1}^{n-k} w_{j,k}^2 R_{j,k}}{\left(\sum_{j=1}^{n-k} w_{j,k}R_{j,k}\right)^2}. \tag{3.5}$$

Proof: Using (3.1), (3.2) and the definition (2.7) of f_k, formula (3.3) can be easily verified.

The conditional unbiasedness of the given estimators is a direct consequence of

$$E\left[\widehat{\alpha}_k|\mathcal{D}_k\right]=\frac{\sum_{i=1}^{n-k} w_{i,k}E\left[S_{i,k+1}|\mathcal{D}_k\right]}{\sum_{i=1}^{n-k} w_{i,k}R_{i,k}}=\frac{\sum_{i=1}^{n-k} w_{i,k}\alpha_k R_{i,k}}{\sum_{i=1}^{n-k} w_{i,k}R_{i,k}}=\alpha_k$$

and similar calculations for $\widehat{\beta}_k$.

Moreover, for $k>l$ we get

$$\begin{aligned}
E\left[\begin{pmatrix}\widehat{\alpha}_k\\ \widehat{\beta}_k\\ \widehat{f}_k\end{pmatrix}\begin{pmatrix}\widehat{\alpha}_l\\ \widehat{\beta}_l\\ \widehat{f}_l\end{pmatrix}^t\middle|\mathcal{D}_l\right] &= E\left[E\left[\begin{pmatrix}\widehat{\alpha}_k\\ \widehat{\beta}_k\\ \widehat{f}_k\end{pmatrix}\middle|\mathcal{D}_k\right]\begin{pmatrix}\widehat{\alpha}_l\\ \widehat{\beta}_l\\ \widehat{f}_l\end{pmatrix}^t\middle|\mathcal{D}_l\right]\\
&= \begin{pmatrix}\alpha_k\\ \beta_k\\ f_k\end{pmatrix}E\left[\begin{pmatrix}\widehat{\alpha}_l\\ \widehat{\beta}_l\\ \widehat{f}_l\end{pmatrix}^t\middle|\mathcal{D}_l\right]\\
&= E\left[\begin{pmatrix}\widehat{\alpha}_k\\ \widehat{\beta}_k\\ \widehat{f}_k\end{pmatrix}\middle|\mathcal{D}_l\right]E\left[\begin{pmatrix}\widehat{\alpha}_l\\ \widehat{\beta}_l\\ \widehat{f}_l\end{pmatrix}^t\middle|\mathcal{D}_l\right],
\end{aligned}$$

which proves (3.4).

Finally, in order to prove (3.5) we will only compute $\mathrm{Var}[\widehat{\alpha}_k|\mathcal{D}_k]$. The other covariances can be obtained in the same way. We get

$$\mathrm{Var}[\widehat{\alpha}_k|\mathcal{D}_k]=\frac{\sum_{j=1}^{n-k} w_{i,k}^2\mathrm{Var}[S_{j,k+1}|\mathcal{D}_k]}{\left(\sum_{j=1}^{n-k} w_{j,k}R_{j,k}\right)^2}=\sigma_k^2\frac{\sum_{j=1}^{n-k} w_{i,k}^2 R_{j,k}}{\left(\sum_{j=1}^{n-k} w_{j,k}R_{j,k}\right)^2}.$$

□

Now we can look at the random variables $S_{i,k}$, $T_{i,k}$ and $R_{i,k}$, for $i+k>n+1$.

Theorem 3.2 *Let the assumptions of Section 2 be fulfilled. Then*

$$\widehat{S}_{i,k} := \widehat{\alpha}_{k-1}\widehat{R}_{i,k-1} = \widehat{\alpha}_{k-1}R_{i,n+1-i}\prod_{l=n+1-i}^{k-2}\widehat{f}_l\,, \tag{3.6}$$

$$\widehat{T}_{i,k} := \widehat{\beta}_{k-1}\widehat{R}_{i,k-1} = \widehat{\beta}_{k-1}R_{i,n+1-i}\prod_{l=n+1-i}^{k-2}\widehat{f}_l\,, \tag{3.7}$$

$$\widehat{R}_{i,k} := R_{i,n+1-i}\prod_{l=n+1-i}^{k-1}\widehat{f}_l \tag{3.8}$$

$$\tag{3.9}$$

are conditionally unbiased estimators for $E[S_{i,k}|\mathcal{D}_{n+1-i}]$, $E[T_{i,k}|\mathcal{D}_{n+1-i}]$ *and* $E[R_{i,k}|\mathcal{D}_{n+1-i}]$, *for* $i+k>n+1$, *respectively.*

Moreover, if there is no tail development, that means $R_{1,n}=0$, *then*

$$\sum_{k=n+2-i}^{n}\widehat{S}_{i,k} = R_{i,n+1-i} + \sum_{k=n+2-i}^{n}\widehat{T}_{i,k}\,. \tag{3.10}$$

Proof: The unbiasedness of the estimators $\widehat{S}_{i,k}$, $\widehat{T}_{i,k}$ and $\widehat{R}_{i,k}$ are direct consequences of Proposition 3.1 and Corollary 2.2.

Now, let us assume that $R_{1,n}=0$. Then we get $\widehat{f}_{n-1}=0$, which yields $\widehat{R}_{i,n}=0$, for $2\le i\le n$. Therefore, it follows

$$\begin{aligned}\sum_{k=n+2-i}^{n}(\widehat{S}_{i,k}-\widehat{T}_{i,k}) &= R_{i,n+1-i}-\widehat{R}_{i,n+2-i}+\sum_{k=n+3-i}^{n}(\widehat{R}_{i,k-1}-\widehat{R}_{i,k})\\ &= R_{i,n+1-i}-\widehat{R}_{i,n}\\ &= R_{i,n+1-i}\,,\end{aligned}$$

this completes the proof.

□

Equation (3.10) shows that there is no gap between the estimations of the total necessary reserves using payments $S_{i,k}$ and using reported amounts $T_{i,k}$. This is an advantage over the standard Chain Ladder method, where one may have a systematically gap between the estimator using payments and the one using reported amounts, see Braun [1]. Therefore, the presented extension of the Complementary Loss Ratio method is in this respect comparable with the Munich Chain Ladder method, which extends the Chain Ladder method in order to eliminate this gap, see Mack and Quarg [4]. Unfortunately, for the Munich Chain Ladder method an estimator for the mean squared error of the estimated necessary reserves is still missing.

4 Estimator of the mean squared error

In this section we want to derive an estimator for the conditional mean squared error

$$\mathrm{mse}(\widehat{R}_i) := E\left[\left(\sum_{k=n+2-i}^{n}(S_{i,k}-\widehat{S}_{i,k})\right)^2\Big|\mathcal{D}_n\right]. \tag{4.1}$$

of the estimated necessary reserves

$$\widehat{R}_i := \sum_{k=n+2-i}^{n}\widehat{S}_{i,n}$$

under the knowledge $\mathcal{D}_n$ of the whole run-off triangle, see also (3.10).

We want to skip problems of dealing with tail factors. That's why we assume

$$R_{1,n}=0. \tag{4.2}$$

In order to get further on we need estimators for σ_k^2 and τ_k^2, for $1\le k<n$, and for γ_k, for $1\le k<n-1$. We take the following weighted conditionally unbiased estimators:

$$\widehat{\sigma}_k^2 := Z_k^{-1}\sum_{i=1}^{n-k}w_{i,k}R_{i,k}\left(\frac{S_{i,k+1}}{R_{i,k}}-\widehat{\alpha}_k\right)^2, \tag{4.3}$$

$$\widehat{\tau}_k^2 := Z_k^{-1}\sum_{i=1}^{n-k}w_{i,k}R_{i,k}\left(\frac{T_{i,k+1}}{R_{i,k}}-\widehat{\beta}_k\right)^2, \tag{4.4}$$

$$\widehat{\gamma}_k := Z_k^{-1}\sum_{i=1}^{n-k}w_{i,k}R_{i,k}\left(\frac{S_{i,k+1}}{R_{i,k}}-\widehat{\alpha}_k\right)\left(\frac{T_{i,k+1}}{R_{i,k}}-\widehat{\beta}_k\right), \tag{4.5}$$

where

$$Z_k := \sum_{i=1}^{n-k}w_{i,k}-\frac{\sum_{i=1}^{n-k}w_{i,k}^2R_{i,k}}{\sum_{i=1}^{n-k}w_{i,k}R_{i,k}},$$

for $1\le k<n-1$. Note that $E[\widehat{\sigma}_k^2|\mathcal{D}_k]=\sigma_k^2$ and analogously for $\widehat{\tau}_k^2$ and $\widehat{\gamma}_k^2$

We still lack estimators for σ_{n-1}^2 and τ_{n-1}^2. If there is even no development after year $n-1$ we set

$$\widehat{\sigma}_{n-1}^2=\widehat{\tau}_{n-1}^2=0.$$

Otherwise, one can extrapolate $\widehat{\sigma}^2_{n-1}$ and $\widehat{\tau}^2_{n-1}$ in the same way as described by Mack [2]. In our examples we take

$$\begin{aligned}\widehat{\sigma}^2_{n-1} &:= \min\left(\frac{\widehat{\sigma}^4_{n-2}}{\widehat{\sigma}^2_{n-3}}, \widehat{\sigma}^2_{n-3}, \widehat{\sigma}^2_{n-2}\right),\\ \widehat{\tau}^2_{n-1} &:= \min\left(\frac{\widehat{\tau}^4_{n-2}}{\widehat{\tau}^2_{n-3}}, \widehat{\tau}^2_{n-3}, \widehat{\tau}^2_{n-2}\right).\end{aligned}$$

Now, fix $n+1-i<k\le n$ and let

$$\mathcal{B}^*_i := \mathcal{B}_{i,n+1-i}\,.$$

Then we obtain

$$\begin{aligned}&E\left[\left(\sum_{k=n+2-i}^{n}(S_{i,k}-\widehat{S}_{i,k})\right)^2\Big|\mathcal{D}_n\right] \qquad (4.6)\\ &= \mathrm{Var}\left[\sum_{k=n+2-i}^{n} S_{i,k}\Big|\mathcal{D}_n\right] + \left(\sum_{k=n+2-i}^{n}(E[S_{i,k}|\mathcal{D}_n]-\widehat{S}_{i,k})\right)^2\\ &= \mathrm{Var}\left[\sum_{k=n+2-i}^{n} S_{i,k}\Big|\mathcal{B}^*_i\right] + \left(\sum_{k=n+2-i}^{n}(E[S_{i,k}|\mathcal{B}^*_i]-\widehat{S}_{i,k})\right)^2\\ &= \sum_{k_1,k_2=n+2-i}^{n}\mathrm{Cov}[S_{i,k_1},S_{i,k_2}|\mathcal{B}^*_i] + \sum_{k_1,k_2=n+2-i}^{n}(E[S_{i,k_1}|\mathcal{B}^*_i]-\widehat{S}_{i,k_1})(E[S_{i,k_2}|\mathcal{B}^*_i]-\widehat{S}_{i,k_2}).\end{aligned}$$

Note, the first term corresponds to the process variance and the second term to the parameter estimation error, see, e.g., Wüthrich-Merz [5].

We split the analysis of the addends of the first sum of the last line into the two parts: $k_1=k_2$ and $k_1<k_2$. For $k:=k_1=k_2$ we get

$$\begin{aligned}\mathrm{Var}[S_{i,k}|\mathcal{B}^*_i] &= E\left[\mathrm{Var}[S_{i,k}|\mathcal{B}_{i,k-1}]\Big|\mathcal{B}^*_i\right] + \mathrm{Var}\left[E[S_{i,k}|\mathcal{B}_{i,k-1}]\Big|\mathcal{B}^*_i\right]\\ &= \sigma^2_{k-1}E[R_{i,k-1}|\mathcal{B}^*_i] + \alpha^2_{k-1}\mathrm{Var}[R_{i,k-1}|\mathcal{B}^*_i].\end{aligned}$$

And for $k_1 < k_2$ we compute

$$
\begin{aligned}
&\mathrm{Cov}[S_{i,k_1}, S_{i,k_2}|\mathcal{B}_i^*] \\
&= E\Big[\mathrm{Cov}[S_{i,k_1}, S_{i,k_2}|\mathcal{B}_{i,k_2-1}]\Big|\mathcal{B}_i^*\Big] + \mathrm{Cov}\Big[E[S_{i,k_1}|\mathcal{B}_{i,k_2-1}], E[S_{i,k_2}|\mathcal{B}_{i,k_2-1}]\Big|\mathcal{B}_i^*\Big] \\
&= \alpha_{k_2-1}\mathrm{Cov}[S_{i,k_1}, R_{i,k_2-1}|\mathcal{B}_i^*] \\
&\vdots \\
&= \alpha_{k_2-1} f_{k_2-2}\cdots f_{k_1}\mathrm{Cov}[S_{i,k_1}, R_{i,k_1}|\mathcal{B}_i^*] \\
&= \alpha_{k_2-1} f_{k_2-2}\cdots f_{k_1}\mathrm{Cov}[S_{i,k_1}, R_{i,k_1-1} - S_{i,k_1} + T_{i,k_1}|\mathcal{B}_i^*] \\
&= \alpha_{k_2-1} f_{k_2-2}\cdots f_{k_1}\Big(\alpha_{k_1-1} f_{k_1-1}\mathrm{Var}[R_{i,k_1-1}|\mathcal{B}_i^*] + (\gamma_{k_1-1} - \sigma_{k_1-1}^2)E[R_{i,k_1-1}|\mathcal{B}_i^*]\Big).
\end{aligned}
$$

Because of Corollary 2.2, we can use the same arguments as in Mack [2] and get

$$
\begin{aligned}
E[R_{i,k}|\mathcal{B}_i^*] &= R_{i,n+1-i}\prod_{l=n+1-i}^{k-1} f_l\,, \\
\mathrm{Var}[R_{i,k}|\mathcal{B}_i^*] &= R_{i,n+1-i}\sum_{l=n+1-i}^{k-1} f_{n+1-i}\cdot\ldots\cdot f_{l-1}(\sigma_l^2 - 2\gamma_l + \tau_l^2) f_{l+1}^2\cdot\ldots\cdot f_{k-1}^2,
\end{aligned}
$$

for all $k > n+1-i$.

Using the notation

$$
\widehat{a}_{k_1,k_2,l} := \begin{cases} \dfrac{\widehat{\sigma}_l^2}{\widehat{\alpha}_l^2}, & \text{for } k_1 = k_2 = l+1, \\ \dfrac{\widehat{\gamma}_l - \widehat{\sigma}_l^2}{\widehat{\alpha}_l \widehat{f}_l}, & \text{for } k_1 > k_2 = l+1 \text{ or } k_2 > k_1 = l+1, \\ \dfrac{\widehat{\sigma}_l^2 - 2\widehat{\gamma}_l + \widehat{\tau}_l^2}{\widehat{f}_l^2}, & \text{for } k_1 \geq k_2 > l+1 \text{ or } k_2 \geq k_1 > l+1, \end{cases} \tag{4.7}
$$

and replacing all unknown variables by their estimators we get for the addends of the first sum of the last line of (4.6)

$$
\widehat{\mathrm{Cov}}[S_{i,k_1}, S_{i,k_2}|\mathcal{B}_i^*] = \widehat{S}_{i,k_1}\widehat{S}_{i,k_2}\sum_{l=n+1-i}^{(k_1\wedge k_2)-1}\frac{\widehat{a}_{k_1,k_2,l}}{\widehat{R}_{i,l}},
$$

where we used the notation $\widehat{R}_{i,k} := R_{i,k}$, for $i + k \leq n+1$.

In order to estimate the addends of the second sum of the last line of (4.6) we start with

$$
\Big(E[S_{i,k}|\mathcal{B}_i^*] - \widehat{S}_{i,k}\Big) = R_{i,n+1-i}\left(\alpha_{k-1}\prod_{l=n+1-i}^{k-2} f_l \quad - \quad \widehat{\alpha}_{k-1}\prod_{l=n+1-i}^{k-2}\widehat{f}_l\right).
$$

Using the abbreviation

$$F_{i,k,l} := \begin{cases} (\alpha_{k-1} - \widehat{\alpha}_{k-1})\widehat{f}_{k-2} \cdot \ldots \cdot \widehat{f}_{n+1-i}, & \text{for } l = k-1, \\ \alpha_{k-1} f_{k-2} \cdot \ldots \cdot f_{l+1}(f_l - \widehat{f}_l)\widehat{f}_{l-1} \cdot \ldots \cdot \widehat{f}_{n+1-i}, & \text{otherwise }, \end{cases} \tag{4.8}$$

we can proceed with

$$\Big(E[S_{i,k_1}|\mathcal{B}_i^*] - \widehat{S}_{i,k_1}\Big)\Big(E[S_{i,k_2}|\mathcal{B}_i^*] - \widehat{S}_{i,k_2}\Big)$$
$$= R_{i,n+1-i}^2 \sum_{l_1=n+1-i}^{k_1-1} \sum_{l_2=n+1-i}^{k_2-1} F_{i,k_1,l_1} F_{i,k_2,l_2}.$$

Now as in Mack [2] we approximate $F_{i,k_1,l_1} F_{i,k_2,l_2}$ by $E[F_{i,k_1,l_1} F_{i,k_2,l_2}|\mathcal{D}_{l_1 \vee l_2}]$. This means we take the average over as little data as possible keeping as much as possible values $S_{i,k}$ and $T_{i,k}$ from the observed data fixed. Using the same arguments as in the proof of Proposition 3.1 we get

$$E[F_{i,k_1,l_1} F_{i,k_2,l_2}|\mathcal{D}_{l_1 \vee l_2}] = 0, \qquad \text{for } l_1 \neq l_2,$$

and

$$E[F_{i,k_1,l} F_{i,k_2,l}|\mathcal{D}_l]$$
$$= \begin{cases} \mathrm{Var}[\widehat{\alpha}_{k_1-1}|\mathcal{D}_{k_1-1}]\widehat{f}_{k_1-2}^2 \cdots \widehat{f}_{n+1-i}^2, & \text{for } k_1 = k_2 = l+1, \\ \alpha_{k_1-1}^2 f_{k_1-2}^2 \cdots f_{l+1}^2 \mathrm{Var}[\widehat{f}_l|\mathcal{D}_l]\widehat{f}_{l-1}^2 \cdots \widehat{f}_{n+1-i}^2, & \text{for } k_1 = k_2 > l+1, \\ \alpha_{k_1-1} f_{k_1-2} \cdots f_{k_2} \mathrm{Cov}[\widehat{f}_{k_2-1}, \widehat{\alpha}_{k_2-1}|\mathcal{D}_{k_2-1}]\widehat{f}_{k_2-2}^2 \cdots \widehat{f}_{n+1-i}^2, & \\ & \text{for } k_1 > k_2 = l+1, \\ \alpha_{k_1-1} f_{k_1-2} \cdots f_{k_2-1}\alpha_{k_2-1} f_{k_2-2}^2 \cdots f_{l+1}^2 \mathrm{Var}[\widehat{f}_l|\mathcal{D}_l]\widehat{f}_{l-1}^2 \cdots \widehat{f}_{n+1-i}^2, & \\ & \text{for } k_1 > k_2 > l+1. \end{cases}$$

Now we use (3.5) and definition (4.7) and replace all unknown variables by their estimators. This and similar arguments for $\widehat{T}_{i,n}$ with the definition

$$\widehat{b}_{k_1,k_2,l} := \begin{cases} \dfrac{\widehat{\tau}_l^2}{\widehat{\beta}_l^2}, & \text{for } k_1 = k_2 = l+1, \\ \dfrac{\widehat{\tau}_l^2 - \widehat{\gamma}_l}{\widehat{\beta}_l \widehat{f}_l}, & \text{for } k_1 > k_2 = l+1 \text{ or } k_2 > k_1 = l+1, \\ \dfrac{\widehat{\sigma}_l^2 - 2\widehat{\gamma}_l + \widehat{\tau}_l^2}{\widehat{f}_l^2}, & \text{for } k_1 \geq k_2 > l+1 \text{ or } k_2 \geq k_1 > l+1, \end{cases}$$

yield

Estimator 4.1 *Let the assumptions of Section 2 be fulfilled. Then the conditional mean squared errors of the estimated necessary reserves $\widehat{R}_i$ and of the estimated IBNR*

$$\widehat{I}_i := \sum_{k=n+2-i}^{n} \widehat{T}_{i,k}$$

can be estimated, respectively, by

$$\widehat{\text{mse}}(\widehat{R}_i) := \sum_{k_1,k_2=n+2-i}^{n} \widehat{S}_{i,k_1} \widehat{S}_{i,k_2} \sum_{l=n+1-i}^{(k_1 \wedge k_2)-1} \widehat{a}_{k_1,k_2,l} \left(\frac{1}{\widehat{R}_{i,l}} + \frac{\sum_{j=1}^{n-l} w_{j,l}^2 R_{j,l}}{\left(\sum_{j=1}^{n-l} w_{j,l} R_{j,l} \right)^2} \right),$$

$$\widehat{\text{mse}}(\widehat{I}_i) := \sum_{k_1,k_2=n+2-i}^{n} \widehat{T}_{i,k_1} \widehat{T}_{i,k_2} \sum_{l=n+1-i}^{(k_1 \wedge k_2)-1} \widehat{b}_{k_1,k_2,l} \left(\frac{1}{\widehat{R}_{i,l}} + \frac{\sum_{j=1}^{n-l} w_{j,l}^2 R_{j,l}}{\left(\sum_{j=1}^{n-l} w_{j,l} R_{j,l} \right)^2} \right).$$

Corollary 4.2 *Let the assumptions of Section 2 be fulfilled and assume that there does not exist any tail development. Then*

$$\widehat{\text{mse}}(\widehat{R}_i) \qquad \textit{and} \qquad \widehat{\text{mse}}(\widehat{I}_i)$$

are estimators for the conditional mean squared error of the ultimate reserves

$$\text{mse}(\widehat{R}_i).$$

Proof: Since we assume that there does not exists any tail development, it follows from Theorem 3.2 that

$$\widehat{R}_i = \sum_{k=n+1-i}^{n} \widehat{S}_{i,k} = R_{i,n+1-i} + \sum_{k=n+1-i}^{n} \widehat{T}_{i,k} = R_{i,n+1-i} + \widehat{I}_i \,.$$

This and the measurability of $R_{i,n+1-i}$ with respect to $\mathcal{D}_n$ proves

$$\text{mse}(\widehat{R}_i) = \text{mse}(\widehat{I}_i),$$

which implies our statement.

□

Often an estimator for the mean squared error of the estimated total necessary reserve

$$\widehat{R} := \sum_{i=2}^{n} \widehat{R}_i$$

is of interest, too. Since $\widehat{R}_i$ and $\widehat{R}_j$, $i \neq j$, depend on the common parameter estimates $\widehat{\alpha}_k$ and $\widehat{\beta}_k$, we cannot simply take the sum of all $\widehat{\text{mse}}(\widehat{R}_i)$. But using the same techniques as for the derivation of the result of Estimator 4.1 we obtain:

Estimator 4.3 *Assume the assumptions of Section 2 are fulfilled. Then the conditional overall mean squared error of the reserves can be estimated by*

$$\widehat{\text{mse}}(\sum_{i=2}^{n} \widehat{R}_i) = \sum_{i=2}^{n} \widehat{\text{mse}}(\widehat{R}_i) + 2 \sum_{2 \leq i_1 < i_2 \leq n} \sum_{k_1=n+2-i_1}^{n} \sum_{k_2=n+2-i_2}^{n} \widehat{S}_{i_1,k_1} \widehat{S}_{i_2,k_2} \sum_{l=n+1-i_1}^{(k_1 \wedge k_2)-1} \widehat{a}_{k_1,k_2,l} \frac{\sum_{j=1}^{n-l} w_{j,l}^2 R_{j,l}}{\Big(\sum_{j=1}^{n-l} w_{j,l} R_{j,l}\Big)^2}.$$

Moreover, the conditional overall mean squared error of the IBNR can be estimated by

$$\widehat{\text{mse}}(\sum_{i=2}^{n} \widehat{I}_i) = \sum_{i=2}^{n} \widehat{\text{mse}}(\widehat{I}_i) + 2 \sum_{2 \leq i_1 < i_2 \leq n} \sum_{k_1=n+2-i_1}^{n} \sum_{k_2=n+2-i_2}^{n} \widehat{T}_{i_1,k_1} \widehat{T}_{i_2,k_2} \sum_{l=n+1-i_1}^{(k_1 \wedge k_2)-1} \widehat{b}_{k_1,k_2,l} \frac{\sum_{j=1}^{n-l} w_{j,l}^2 R_{j,l}}{\Big(\sum_{j=1}^{n-l} w_{j,l} R_{j,l}\Big)^2}.$$

Derivation: We start with

$$\text{mse}(\sum_{i=2}^{n} \widehat{R}_i) = \sum_{i=2}^{n} \text{mse}(\widehat{R}_i) + 2 \sum_{2 \leq i_1 < i_2 \leq n} \sum_{k_1=n+2-i_1}^{n} \sum_{k_2=n+2-i_2}^{n} \Big(E[S_{i_1,k_1}|\mathcal{D}_n] - \widehat{S}_{i_1,k_1}\Big)\Big(E[S_{i_2,k_2}|\mathcal{D}_n] - \widehat{S}_{i_2,k_2}\Big).$$

Similar to the derivation of Estimator 4.1 we substitute

$$E[S_{i,k}|\mathcal{D}_n] - \widehat{S}_{i,k} = \sum_{l=n+1-i}^{k-1} F_{i,k,l},$$

where $F_{i,k,l}$ is defined by (4.8). This leads to

$$\text{mse}(\sum_{i=2}^{n} \widehat{R}_i) = \sum_{i=2}^{n} \text{mse}(\widehat{R}_i)$$
$$+ 2 \sum_{2 \le i_1 < i_2 \le n} \sum_{k_1=n+2-i_1}^{n} \sum_{k_2=n+2-i_2}^{n} \sum_{l_1=n+1-i_1}^{k_1-1} \sum_{l_2=n+1-i_2}^{k_2-1} F_{i_1,k_1,l_1} F_{i_2,k_2,l_2}.$$

We proceed with the product $F_{i_1,k_1,l_1} F_{i_2,k_2,l_2}$ in the same way as with $F_{i,k_1,l_1} F_{i,k_2,l_2}$ in the derivation of Estimator 4.1 and obtain our claimed estimator for the overall mean squared error of the reserves by replacing all unknown parameters with their estimators.

Moreover, the same procedure leads to the claimed estimator for the overall mean squared error of the IBNR.

□

Finally, the same arguments as in the proof of Corollary 4.2 yield

Corollary 4.4 *Let the assumptions of Section 2 be fulfilled and assume that there does not exist any tail development. Then*

$$\widehat{\text{mse}}(\sum_{i=2}^{n} \widehat{R}_i) \qquad \textit{and} \qquad \widehat{\text{mse}}(\sum_{i=2}^{n} \widehat{I}_i)$$

are estimators for the conditional overall mean squared error of the ultimate reserves

$$\text{mse}(\sum_{i=2}^{n} \widehat{R}_i).$$

Note, there are also other methods that lead to an estimate for the estimation error (second term in (4.6)). An alternative would be to apply the conditional resampling approach, see Wüthrich-Merz [5].

5 Two examples

As first example we choose some complete triangles and no weights in order to compare the standard Chain Ladder method, see Mack [2], with our method. The triangles are shown in Table 4 and Table 5. As estimators for the parameters we get the values shown in Table 1, Table 2 and Table 3.

k	1	2	3	4	5	6	7	8	9
$\widehat{f}_k$	1.2343	1.2904	1.1918	1.1635	1.1457	1.1013	1.0702	1.0760	1.0444
$\widehat{\sigma}_k^2$	6'658	9'884	8'707	1'497	2'321	5'522	1'850	8'024	1'850

Table 1: Chain Ladder parameters for payments of Example 1

k	1	2	3	4	5	6	7	8	9
$\widehat{f}_k$	1.6502	0.8561	0.8718	0.9614	0.9812	0.9833	0.9900	0.9917	0.9949
$\widehat{\sigma}_k^2$	31'586	7'885	5'771	538	235	10	13	4	1

Table 2: Chain Ladder parameters for reported amounts of Example 1

k	1	2	3	4	5	6	7	8	9
$\widehat{\alpha}_k$	0.1174	0.0922	0.1114	0.1764	0.2424	0.3002	0.3271	0.4279	0.8923
$\widehat{\beta}_k$	0.9761	-0.1896	-0.2026	-0.0802	-0.0501	-0.0663	-0.0564	-0.0548	-0.1077
$\widehat{\sigma}_k^2$	4'241	5'560	5'103	2'796	16'724	9'625	18'536	26	0
$\widehat{\tau}_k^2$	48'855	10'044	11'535	856	300	1'025	567	345	210
$\widehat{\gamma}_k$	1'931	2'771	1'403	-175	-47	-895	-3'130	-95	

Table 3: Extended Complementary Loss Ratio parameters of Example 1

The resulting estimated reserves are shown in Table 6. Table 7 and Table 8 list the corresponding standard errors and the standard errors in percent of the reserves. In this example the extended Complementary Loss Ratio method leads to reserves which lie for almost all accident years between the two corresponding Chain Ladder reserves.

	Chain Ladder		Ext. Compl.
	Paid	Incurred	Loss Ratio
1	0	0	0
2	114'086	337'984	314'902
3	394'121	31'884	66'994
4	608'749	331'436	359'384
5	697'742	1'018'350	981'883
6	1'234'157	1'103'928	1'115'768
7	1'138'623	1'868'664	1'786'947
8	1'638'793	1'997'651	1'942'518
9	2'359'939	1'418'779	1'569'657
10	1'979'401	2'556'612	2'590'718
Total	10'165'612	10'665'287	10'728'771

Table 6: Estimated reserves of Example 1

	1	2	3	4	5	6	7	8	9	10
1	1'216'632	1'347'072	1'786'877	2'281'606	2'656'224	2'909'307	3'283'388	3'587'549	3'754'403	3'921'258
2	798'924	1'051'912	1'215'785	1'349'939	1'655'312	1'926'210	2'132'833	2'287'311	2'567'056	
3	1'115'636	1'387'387	1'930'867	2'177'002	2'513'171	2'931'930	3'047'368	3'182'511		
4	1'052'161	1'321'206	1'700'132	1'971'303	2'298'349	2'645'113	3'003'425			
5	808'864	1'029'523	1'229'626	1'590'338	1'842'662	2'150'351				
6	1'016'862	1'251'420	1'698'052	2'105'143	2'385'339					
7	948'312	1'108'791	1'315'524	1'487'577						
8	917'530	1'082'426	1'484'405							
9	1'001'238	1'376'124								
10	841'930									

Table 4: Cumulated triangle of payments of Example 1

	1	2	3	4	5	6	7	8	9	10
1	3'362'115	5'217'243	4'754'900	4'381'677	4'136'883	4'094'140	4'018'736	3'971'591	3'941'391	3'921'258
2	2'640'443	4'643'860	3'869'954	3'248'558	3'102'002	3'019'980	2'976'064	2'946'941	2'919'955	
3	2'879'697	4'785'531	4'045'448	3'467'822	3'377'540	3'341'934	3'283'928	3'257'827		
4	2'933'345	5'299'146	4'451'963	3'700'809	3'553'391	3'469'505	3'413'921			
5	2'768'181	4'658'933	3'936'455	3'512'735	3'385'129	3'298'998				
6	3'228'439	5'271'304	4'484'946	3'798'384	3'702'427					
7	2'927'033	5'067'768	4'066'526	3'704'113						
8	3'083'429	4'790'944	4'408'097							
9	2'761'163	4'132'757								
10	3'045'376									

Table 5: Cumulated triangle of reported amounts of Example 1

	Chain Ladder		Ext. Compl. Loss Ratio	
	Paid	Incurred	Paid	Incurred
1				
2	89'423	2'553	194	14'639
3	234'652	5'186	4'557	5'538
4	255'590	9'264	10'541	12'566
5	261'272	10'874	36'792	38'250
6	323'859	33'243	43'940	44'835
7	274'914	55'884	65'055	65'909
8	373'587	165'086	176'706	176'977
9	492'815	209'162	197'781	197'917
10	468'074	321'560	322'900	323'049
Total	1'517'480	455'794	467'814	471'873

Table 7: Estimated standard errors of the reserves of Example 1

	Chain Ladder		Ext. Compl. Loss Ratio	
	Paid	Incurred	Paid	Incurred
1				
2	78.4%	0.8%	0.1%	4.6%
3	59.5%	16.3%	6.8%	8.3%
4	42.0%	2.8%	2.9%	3.5%
5	37.4%	1.1%	3.7%	3.9%
6	26.2%	3.0%	3.9%	4.0%
7	24.1%	3.0%	3.6%	3.7%
8	22.8%	8.3%	9.1%	9.1%
9	20.9%	14.7%	12.6%	12.6%
10	23.6%	12.6%	12.5%	12.5%
Total	14.9%	4.3%	4.4%	4.4%

Table 8: Estimated standard errors in percent of the reserves of Example 1

It is remarkable that although we have to estimate 44 parameters for the extended Complementary Loss Ration method compared to 18 for the Chain Ladder method, we don't get much higher and for some years even smaller standard errors. This may be caused by the estimated negative correlations $\widehat{\gamma}_k$, for $k > 3$. And because you combine two sources of information, that is, you have more parameters to estimate but you also have more observations to do this parameter estimation.

	1	2	3	4	5	6	7	8	9	10
1						609'570	586'185	428'927	251'449	393'286
2					923'457	593'791	227'078	92'196	237'614	
3				2'088'845	1'078'031	1'218'159	608'392	188'167		
4			4'308'763	1'554'327	936'982	685'698	611'920			
5		7'768'561	5'135'659	1'822'280	972'849	776'484				
6	552'074	8'531'821	5'170'716	2'002'110	832'309					
7	541'244	8'743'124	5'597'618	2'441'332						
8	642'910	10'880'562	6'204'158							
9	612'030	10'560'382								
10	843'267									

Table 9: Incremental triangle of payments of Example 2

	1	2	3	4	5	6	7	8	9	10
1						-1'300'429	165'398	-449'931	-375'029	-202'553
2					-605'101	-320'520	-318'715	3'429	-475'913	
3				-305'795	-84'808	-287'127	-1'111'628	-1'142'698		
4			2'951'080	-1'206'168	-573'665	-863'962	-734'588			
5		16'963'154	6'306'204	-629'316	-1'127'533	38'569				
6	1'216'693	19'307'493	4'610'965	119'541	-907'170					
7	2'099'009	24'019'545	5'522'897	-2'558'489						
8	2'014'710	27'618'640	4'469'660							
9	2'339'252	28'149'520								
10	2'413'472									

Table 10: Incremental triangle of reported amounts of Example 2

For the second example we take some part of a motor liability portfolio. We took all small bodily injury claims. Because of the lack of good information, we can trust the split into bodily injury and property damage claims only for the last 6 business years. The corresponding incremental triangles are shown in Table 9 and Table 10. The opening reserves $R_{i,6-i}$, $1 \leq i \leq 5$, are given by Table 11

i	$R_{i,6-i}$
1	5'210'174
2	6'093'211
3	11'142'105
4	11'467'760
5	887'953

Table 11: Opening reserves for Example 2

We choose weights

$$w_{i,k} := \begin{cases} 0, & \text{for } i+k \leq 5, \\ 1, & \text{otherwise.} \end{cases}$$

Moreover, since we only see 10 years of claim development, we have to decide how to deal with the reserves remaining after 10 years. In this example we assume that only fifty percent of those remaining reserves will be paid. Note, this decision does not affect the estimated standard errors.

Applying our method we get the following estimators for the parameters:

k	1	2	3	4	5	6	7	8	9
$\widehat{\alpha}_k$	7.4862	0.3889	0.1647	0.1186	0.1299	0.1174	0.0686	0.0975	0.2862
$\widehat{\beta}_k$	18.6909	0.3512	-0.0762	-0.0825	-0.0914	-0.1155	-0.1536	-0.1696	-0.1474
$\widehat{\sigma}_k^2$	7'359'451	71'545	4'301	3'522	2'561	9'217	13'058	2'646	536
$\widehat{\tau}_k^2$	25'224'905	274'131	57'645	17'390	59'029	44'779	62'834	1'058	18
$\widehat{\gamma}_k$	13'351'758	123'550	14'805	1'853	4'527	9'429	-3'633	-1'673	

Table 12: Estimated parameters of Example 2

This leads to the following estimated reserves and their corresponding standard errors (s.e.):

			Paid		Reported Amount	
	Reserves	IBNR	s.e.	in % Res.	s.e.	in % Res.
1	389'107	-389'107				
2	1'310'917	-991'339	57'460	4.4%	10'474	0.8%
3	1'559'034	-1'469'423	82'210	5.3%	45'552	2.9%
4	1'380'074	-1'562'693	211'574	15.3%	351'627	25.5%
5	2'845'519	-3'117'679	424'820	14.9%	635'533	22.3%
6	3'639'882	-3'618'609	513'117	14.1%	769'909	21.2%
7	6'106'104	-5'653'541	664'565	10.9%	969'190	15.9%
8	9'152'283	-7'223'097	943'067	10.3%	1'264'629	13.8%
9	17'901'115	-1'415'244	2'173'399	12.1%	2'486'225	13.9%
10	29'514'639	27'944'434	6'960'209	23.6%	7'413'137	25.1%
Total	73'798'673	2'503'701	7'803'265	10.6%	8'681'194	11.8%

Table 13: Estimated reserves and standard errors of Example 2

If you try to apply the Chain Ladder method to the last 6 years, you would have to estimate a tail factor for the payments of about 1.17 and for the reported amount of about 0.87 in order to get similar results for the total necessary reserves.

Acknowledgement: I would like to thank Benedetto Conti for introducing me to non life insurance and reserving. Working with him pushed me to develop the presented reserving method.

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