CLAIMS DEVELOPMENT RESULT FOR COMBINED CLAIMS INCURRED AND CLAIMS PAID DATA

René DAHMS*

Michael MERZ†

Mario V. WÜTHRICH‡

Abstract

We present the one-year claims development result (CDR) in the complementary loss ratio method (CLRM). The complementary loss ratio method presented in Dahms [3] is a stochastic claims reserving method that considers simulateneously claims paid data and claims incurred data. In this model we study the conditional mean square error of prediction (MSEP) for the one-year claims development result uncertainty. This is an important view in all new solvency considerations and in risk-based controlling of non-life insurance companies.

Keywords: stochastic loss reserving, complementary loss ratio method, ultimate claim, conditional mean square error of prediction, claims development result, claims experience prior accident years.

*Bâloise Insurance, Aeschengraben 21, 4002 Basel, Switzerland

†University Tübingen, Faculty of Economics, 72074 Tübingen, Germany

‡ETH Zurich, Department of Mathematics, 8092 Zurich, Switzerland

1 INTRODUCTION

The main task of actuaries is to predict random variables and future cashflows in an appropriate way. This serves as a basis for liability estimations and premium calculations. For these cashflow predictions one often has different sources of information. A major difficulty is to combine the knowledge from these different sources of information appropriately. Claims reserving is one area where one faces this task. In the present paper we assume that we have claims incurred data (case estimates for reported claims) and claims paid data. It is well-known that claims reserves based on claims incurred data may substantially differ from the claims reserves based on claims paid data. Therefore we try to combine these two sources to get a unified (and more reliable) prediction for the outstanding loss liabilities.

Halliwell [4] is probably one of the first who has investigated this problem from a statistical point of view. Quarg-Mack [9] proposed a successful way to combine claims incurred and claims paid data for claims reserving. They have introduced the so-called Munich chain ladder method which provides predictions for the outstanding loss liabilities that are based both on claims incurred and claims paid data. However, so far, there is no appropriate way to quantify the prediction uncertainty within the Munich chain ladder method. Recently, Dahms [3] has given a different stochastic model that combines these two sources of information. Dahms [3] has extended the complementary loss ratio method (CLRM) for deriving predictions based on claims incurred and claims paid data simultaneously by choosing the case reserves as basis for the regression. This method is successfully applied in various non-life insurance companies that have claims incurred data with a minimal consistency standard over time. Another major advantage of the extended CLRM method is that allows for the derivation of a mean square error of prediction (MSEP) estimate. We revisit Dahms' method [3] within a solvency framework.

In most solvency considerations one is interested into the changes and uncertainties over a one-year time horizon. That is, one predicts the outstanding loss liabilities today and in one year with the new information available in one year. The difference between these two successive predictions is the so-called claims development result (CDR). The CDR is of central interest in every solvency consideration because it corresponds to a profit & loss statement position that directly influences the financial strength of an insurance company. This one-year view and the derivation of the CDR has therefore attracted a lot of attention in recent research, for instance, Merz-Wüthrich [7] analyze the CDR for the distribution-free chain ladder model of Mack [5] or Bühlmann et al. [2] analyze the CDR within the credibility chain ladder model. In the present paper we consider the CDR for the CLRM presented in Dahms [3]. This way, we solve several problems at the same time. Namely, we consider the CDR for solvency purposes in a model where one is able to combine the information from claims incurred and claims paid data. Moreover, the present model gives the predicted cashflow pattern in a natural way, i.e. this model also allows for market-consistent valuation of the liability cashflows if one applies an appropriate deflator or discount function (time values of cashflows).

2 NOTATION AND MODEL DEFINITION

Usually, claims reserving data are studied in so-called claims development triangles (see Figure 1). For accident years we use the index $i \in \{0, \dots, I\}$ and for development years we use the index $j \in \{0, \dots, J\}$. For simplicity, we assume $I = J$. Then we introduce the following notation:

- $C_{i,j}^{Pa}$ denotes cumulative payments for all claims with accident year i up to development year j.
- $C_{i,j}^{In}$ denotes claims incurred (case estimates) for all claims with accident year i reported by the end of development year j.
- $X_{i,j}^{Pa} = C_{i,j}^{Pa} - C_{i,j-1}^{Pa}$ denotes incremental payments within development year j for all claims with accident year i (set $C_{i,-1}^{Pa} = 0$, i.e. $X_{i,0}^{Pa} = C_{i,0}^{Pa}$).
- $X_{i,j}^{In} = C_{i,j}^{In} - C_{i,j-1}^{In}$ denotes the change in claims incurred (case estimates) within

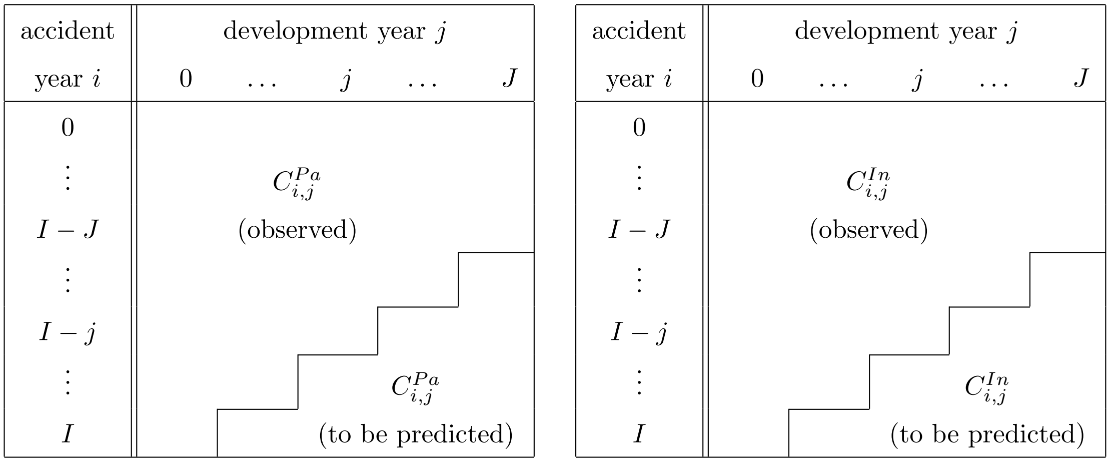

Figure 1: Claims development triangles for cumulative payments and claims incurred.

development year j for all claims with accident year i (set $C_{i,-1}^{In} = 0$, i.e. $X_{i,0}^{In} = C_{i,0}^{In}$).

- Finally, the case reserves for accident year i at the end of development year j are given by

$$R_{i,j} = C_{i,j}^{In} - C_{i,j}^{Pa}. \tag{1}$$

The following recursive formula for case reserves is straightforward

$$R_{i,j} = R_{i,j-1} + X_{i,j}^{In} - X_{i,j}^{Pa}, \qquad \text{for } j = 0, \dots, J, \tag{2}$$

with $R_{i,-1} = 0$. We define the two dimensional random vectors

$$\mathbf{C}_{i,j} = \left(C_{i,j}^{Pa}, C_{i,j}^{In}\right)' \qquad \text{and} \qquad \mathbf{X}_{i,j} = \left(X_{i,j}^{Pa}, X_{i,j}^{In}\right)'. \tag{3}$$

After accounting year I we have observations in the upper triangle (see Figure 1)

$$\mathcal{D}_I = \left\{\mathbf{C}_{i,j};\ 0 \le i \le I,\ 0 \le j \le J,\ i + j \le I\right\}, \tag{4}$$

and after accounting year $I+1$ we have observations in the trapezoid

$$\begin{aligned} \mathcal{D}_{I+1} &= \left\{\mathbf{C}_{i,j};\ 0 \le i \le I,\ 0 \le j \le J,\ i + j \le I + 1\right\} \qquad (5) \\ &= \mathcal{D}_I \ \cup\ \left\{\mathbf{C}_{i,j};\ 0 \le i \le I,\ 0 \le j \le J,\ i + j = I + 1\right\}, \end{aligned}$$

that is, the update $\mathcal{D}_I \mapsto \mathcal{D}_{I+1}$ adds a new diagonal to the observations available at time I. Based on these observations we need to predict the ultimate claims $\mathbf{C}_{i,J}$ at time I and $I+1$, respectively. We define the set of observations up to development year $k \in \{0, \ldots, J\}$ by

$$\mathcal{B}_k = \{\mathbf{C}_{i,j};\ 0 \le i \le I,\ 0 \le j \le k\}. \tag{6}$$

Model Assumptions 2.1 (Extended Complementary Loss Ratio Method)

We assume that the random vectors $(\mathbf{C}_{i,j})_{0\le j\le J}$ are independent for different accident years i. There exists deterministic factors f_j, g_j and positive definit 2×2 covariance matrices Σ_j, $j = 0, \ldots, J-1$, such that

$$E\left[\mathbf{X}_{i,j+1}\middle|\,\mathcal{B}_j\right] = \left(R_{i,j}\ f_j\ ,\ R_{i,j}\ g_j\right)', \tag{7}$$

$$\mathrm{Cov}\left(\mathbf{X}_{i,j+1}, \mathbf{X}_{i,j+1}\middle|\,\mathcal{B}_j\right) = R_{i,j}\ \Sigma_j. \tag{8}$$

Moreover, we assume that $R_{i,J} = 0$, P-a.s.

□

Remarks 2.2

- Dahms [3] has extended the complementary loss ratio method (see Mack [6]) by choosing the outstanding case reserves as the appropriate underlying risk exposure. Nevertheless, we call Dahms' extension CLRM.

- Under Model Assumptions 2.1 the next incremental claims $\mathbf{X}_{i,j+1}$ are regressed from the last case reserves $R_{i,j}$. This is a very common model in practice, especially in liability lines of business. Of course, this requires that the claims incurred estimation is done consistently over time otherwise the model assumptions are not fulfilled (see also last bullet point of these remarks).

- For cumulative claims we obtain

$$\begin{aligned} E\left[\mathbf{C}_{i,j+1}\middle|\,\mathcal{B}_j\right] &= \left(C_{i,j}^{Pa} + R_{i,j} f_j, C_{i,j}^{In} + R_{i,j} g_j\right)' \\ &= \left(f_j C_{i,j}^{In} + (1-f_j)\ C_{i,j}^{Pa}, (1+g_j)\ C_{i,j}^{In} - g_j C_{i,j}^{Pa}\right)'. \end{aligned} \tag{9}$$

From this we see that both claims incurred and claims paid are simultaneously regressed from the last incurred and paid observation. This is different from the classical one-dimensional distribution-free chain ladder model (see Mack [5]) where only one source of information is considered.

- From Corollary 2.2 in Dahms [3] one sees that the case reserves satisfy a chain ladder like assumption, i.e.

$$E\left[R_{i,j+1} \middle| \mathcal{B}_j \right] = R_{i,j}\ (1 + g_j - f_j), \tag{10}$$

$$\mathrm{Var}\left(R_{i,j+1} \middle| \mathcal{B}_j \right) = R_{i,j}\left(\sigma_j^{1,1} - 2\sigma_j^{1,2} + \sigma_j^{2,2} \right), \tag{11}$$

where $\Sigma_j = (\sigma_j^{m,n})_{m,n=1,2}$. We define the chain ladder factor for the case reserves by

$$h_j = 1 + g_j - f_j. \tag{12}$$

- $R_{i,J} = 0$ means that $C_{i,J}^{In} = C_{i,J}^{Pa}$ which we associate with the fact that J is the final development year where every claim is finally settled and closed, and that there is no claims development beyond development year J. This implies that $h_{J-1} = 1 + g_{J-1} - f_{J-1} = 0$.

- Note that implicitly we assume that $R_{i,j} \geq 0$, P-a.s., otherwise the covariance assumptions in Model Assumptions 2.1 are not meaningful. This implies that $C_{i,j}^{In} \geq C_{i,j}^{Pa}$, P-a.s., and that $1 + g_j - f_j \geq 0$.

- Note that the claims reserving method presented here also applies if we have incomplete triangles. This means that this method can also be used if, for example, we do not have the entire history of all claims or if the claims estimation guidelines (for the estimation of claims incurred) has changed at some stage in the past. For the handling of such incomplete triangles we refer to Dahms [3].

3 ULTIMATE CLAIMS PREDICTION AND CDR

3.1 Ultimate Claims Prediction at Time I

In this subsection we assume that we are at time I. We have observations $\mathcal{D}_I$ and we need to predict the ultimate claims $\mathbf{C}_{i,J}$ under Model Assumptions 2.1. Since the parameters f_j, g_j and h_j are, in general, not known they need to be estimated from the data:

$$\widehat{f}_j^I = \frac{\sum_{i=0}^{I-j-1} X_{i,j+1}^{Pa}}{\sum_{i=0}^{I-j-1} R_{i,j}} \qquad \text{and} \qquad \widehat{g}_j^I = \frac{\sum_{i=0}^{I-j-1} X_{i,j+1}^{In}}{\sum_{i=0}^{I-j-1} R_{i,j}}, \tag{13}$$

moreover we set $\widehat{h}_j^I = 1 + \widehat{g}_j^I - \widehat{f}_j^I$.

Note that $\widehat{f}_j^I$, $\widehat{g}_j^I$ and $\widehat{h}_j^I$ are conditionally, given $\mathcal{B}_j$, unbiased estimators for f_j, g_j and h_j, respectively. Moreover, they are uncorrelated for different indices j (see Proposition 3.1 in Dahms [3]) and they satisfy an optimality condition for the second moment similar to Lemma 3.3 in Wüthrich-Merz [10].

Using (10) and (7) and (2) this motivates the following predictors for $i+j>I$ (an empty product is set to 1):

$$\widehat{R}_{i,j}^I = R_{i,I-i} \prod_{k=I-i}^{j-1} \widehat{h}_k^I, \tag{14}$$

$$(\widehat{X}_{i,j}^{Pa})^I = R_{i,I-i} \prod_{k=I-i}^{j-2} \widehat{h}_k^I \, \widehat{f}_{j-1}^I, \tag{15}$$

$$(\widehat{X}_{i,j}^{In})^I = R_{i,I-i} \prod_{k=I-i}^{j-2} \widehat{h}_k^I \, \widehat{g}_{j-1}^I. \tag{16}$$

These are conditionally, given $\mathcal{B}_{I-i}$, unbiased estimators for $E\left[R_{i,j}\middle|\mathcal{D}_I\right]$, $E\left[X_{i,j}^{Pa}\middle|\mathcal{D}_I\right]$ and $E\left[X_{i,j}^{In}\middle|\mathcal{D}_I\right]$, respectively (see Dahms [3], Theorem 3.2) and are therefore used to predict the ultimate claims $\mathbf{C}_{i,J}$ at time I. Hence, at time I we set

$$\widehat{C}_{i,J}^I = C_{i,I-i}^{In} + \sum_{j=I-i+1}^{J} (\widehat{X}_{i,j}^{In})^I = C_{i,I-i}^{Pa} + \sum_{j=I-i+1}^{J} (\widehat{X}_{i,j}^{Pa})^I. \tag{17}$$

The last equality is a consequence of $R_{i,J}=0$, i.e., that the final development year is J (see Dahms [3], equality (3.10)). This implies that we get *one* estimate for the claims reserves based on claims incurred and claims paid data, simultaneously. This is different from the Munich chain ladder method (see Quarg-Mack [9]), which reduces the gap between the estimated claims reserves based on claims paid data and estimated claims reserves based on claims incurred data, but still leads to two different estimates.

The variance parameters are estimated as follows (see Dahms [3] (4.3)-(4.5)), for $j < I-1$

$$\widehat{\sigma}_j^{1,1} = \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} R_{i,j} \left(\frac{X_{i,j+1}^{Pa}}{R_{i,j}} - \widehat{f}_j^I \right)^2, \tag{18}$$

$$\widehat{\sigma}_j^{2,2} = \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} R_{i,j} \left(\frac{X_{i,j+1}^{In}}{R_{i,j}} - \widehat{g}_j^I \right)^2, \tag{19}$$

$$\widehat{\sigma}_j^{1,2} = \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} R_{i,j} \left(\frac{X_{i,j+1}^{Pa}}{R_{i,j}} - \widehat{f}_j^I \right) \left(\frac{X_{i,j+1}^{In}}{R_{i,j}} - \widehat{g}_j^I \right). \tag{20}$$

For $j = I-1$ we do not have enough observations to estimate second moments, therefore we use Mack's [5] estimator for the last variance parameters, see also Dahms [3] or Example 8.21 in Wüthrich-Merz [10] for the explicit formulas.

3.2 Ultimate Claims Prediction at Time $I+1$ and the Observable Claims Development Result

Now, we assume that we are at time $I+1$ and that we have observations $\mathcal{D}_{I+1}$. The parameters are then estimated by

$$\widehat{f}_j^{I+1} = \frac{\sum_{i=0}^{I-j} X_{i,j+1}^{Pa}}{\sum_{i=0}^{I-j} R_{i,j}} \qquad \text{and} \qquad \widehat{g}_j^{I+1} = \frac{\sum_{i=0}^{I-j} X_{i,j+1}^{In}}{\sum_{i=0}^{I-j} R_{i,j}}, \tag{21}$$

moreover we set $\widehat{h}_j^{I+1} = 1 + \widehat{g}_j^{I+1} - \widehat{f}_j^{I+1}$. Henceforth, at time $I+1$ we have the predictors

$$\widehat{R}_{i,j}^{I+1} = R_{i,I-i+1} \prod_{k=I-i+1}^{j-1} \widehat{h}_k^{I+1}, \tag{22}$$

$$(\widehat{X}_{i,j}^{Pa})^{I+1} = R_{i,I-i+1} \prod_{k=I-i+1}^{j-2} \widehat{h}_k^{I+1}\ \widehat{f}_{j-1}^{I+1}, \tag{23}$$

$$(\widehat{X}_{i,j}^{In})^{I+1} = R_{i,I-i+1} \prod_{k=I-i+1}^{j-2} \widehat{h}_k^{I+1}\ \widehat{g}_{j-1}^{I+1}. \tag{24}$$

Therefore, we set at time $I+1$

$$\widehat{C}_{i,J}^{I+1} = C_{i,I-i+1}^{In} + \sum_{j=I-i+2}^{J} (\widehat{X}_{i,j}^{In})^{I+1} = C_{i,I-i+1}^{Pa} + \sum_{j=I-i+2}^{J} (\widehat{X}_{i,j}^{Pa})^{I+1}, \tag{25}$$

which is now a $\mathcal{D}_{I+1}$-measurable random variable.

The observable claims development result (CDR) at time $I+1$ for accident year i is then given by

$$\widehat{\mathrm{CDR}}_i(I+1) = \widehat{C}_{i,J}^{I} - \widehat{C}_{i,J}^{I+1}. \tag{26}$$

This is exactly the position that is observed in the position "claims experience prior accident years" in the profit & loss statement. Therefore, we need to study its uncertainty for solvency purposes. For an extended discussion on the CDR we refer to Ohlsson-Lauzeningks [8] and Merz-Wüthrich [7].

4 MEAN SQUARE ERROR OF PREDICTION OF THE CDR

4.1 Single Accident Years

The uncertainty in the prediction of the outstanding loss liabilities is often studied in terms of the conditional mean square error of prediction (MSEP), see Section 3.1 in Wüthrich-Merz [10]. Since in the budget statement the CDR is usually predicted by 0, we study the uncertainty of this prediction. The conditional MSEP is then defined by

$$\mathrm{msep}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathcal{D}_I}(0) = E\left[\left(\widehat{\mathrm{CDR}}_i(I+1) - 0\right)^2 \middle| \mathcal{D}_I\right]. \tag{27}$$

Pluging in the definition (26) of the observable CDR we obtain

$$\mathrm{msep}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathcal{D}_I}(0) = E\left[\left(\widehat{C}_{i,J}^{I} - \widehat{C}_{i,J}^{I+1}\right)^2 \middle| \mathcal{D}_I\right], \tag{28}$$

that means we need to study the fluctuations of successive ultimate claims predictions. Similar as in Merz-Wüthrich [7] we split the MSEP into process variance term and parameter estimation error term:

$$\mathrm{msep}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathcal{D}_I}(0) = \mathrm{Var}\left(\widehat{\mathrm{CDR}}_i(I+1)\middle| \mathcal{D}_I\right) + E\left[\widehat{\mathrm{CDR}}_i(I+1)\middle| \mathcal{D}_I\right]^2. \tag{29}$$

Note that we could also have other splits as the Bayesian model considered in Bühlmann et al. [2] indicates. The first term on the right-hand side (29) corresponds to the process variance and the second term to the parameter estimation error term. Let us briefly discuss the parameter estimation error term. Note that due to the $\mathcal{D}_I$-measurability of $\widehat{C}^I_{i,J}$ the parameter estimation error term is given by

$$E\left[\widehat{\mathrm{CDR}}_i(I+1)\middle|\mathcal{D}_I\right]^2 = \left(\widehat{C}^I_{i,J} - E\left[\widehat{C}^{I+1}_{i,J}\middle|\mathcal{D}_I\right]\right)^2 \stackrel{def.}{=} R^2_{i,I-i}\ \Delta_i. \tag{30}$$

We use (30) as definition for Δ_i. With Corollary 4.1 below it becomes clear that the last observation on the diagonal $R^2_{i,I-i}$ is the correct scaling for the parameter estimation error term (see also (66) below). Observe that on the one hand we have the predictor given at time I

$$\widehat{C}^I_{i,J} = C^{Pa}_{i,I-i} + R_{i,I-i} \sum_{j=I-i+1}^{J} \prod_{k=I-i}^{j-2} \widehat{h}^I_k\ \ \widehat{f}^I_{j-1}, \tag{31}$$

and on the other hand we need to compare this to

$$E\left[\widehat{C}^{I+1}_{i,J}\middle|\mathcal{D}_I\right] = E\left[C^{Pa}_{i,I-i+1} + R_{i,I-i+1} \sum_{j=I-i+2}^{J} \prod_{k=I-i+1}^{j-2} \widehat{h}^{I+1}_k\ \ \widehat{f}^{I+1}_{j-1}\middle|\mathcal{D}_I\right]. \tag{32}$$

We define

$$\delta_j = \frac{R_{I-j,j}}{\sum_{i=0}^{I-j} R_{i,j}} \in [0,1]. \tag{33}$$

Note that δ_j is $\mathcal{D}_I$-measurable, i.e. observable at time I and it denotes the contribution of the last observed case estimate for a fixed development year j. Lemma 6.1, below, implies for the observable CDR the following corollary.

Corollary 4.1 *Under Model Assumptions 2.1 we have*

$$E\left[\widehat{\mathrm{CDR}}_i(I+1)\middle|\mathcal{D}_I\right] = R_{i,I-i}\left(\widehat{f}^I_{I-i} - f_{I-i}\right) + R_{i,I-i} \sum_{j=I-i+2}^{J} \left[\prod_{k=I-i}^{j-2} \widehat{h}^I_k\ \ \widehat{f}^I_{j-1}\right.$$
$$\left. - h_{I-i} \prod_{k=I-i+1}^{j-2} \left((1-\delta_k)\ \widehat{h}^I_k + \delta_k\ h_k\right)\left((1-\delta_{j-1})\ \widehat{f}^I_{j-1} + \delta_{j-1}\ \ f_{j-1}\right)\right].$$

Corollary 4.1 is a crucial result for the understanding of successive predictions and the observable CDR. Note that in general the parameter estimators deviate from the true parameter values. This shows that the expected observable CDR is in general different from zero and one loses the martingale property. However, the expected observable CDR is predicted by zero in the budget statement. Henceforth, the consideration of the parameter estimation error (last term in (29)) exactly quantifies this deviation and requires some care, as will be seen below.

Note that Corollary 4.1 and Lemma 6.1, below, also hold true if we replace all variables that correspond to claims payments by the parameters that correspond to claims incurred. As in Dahms [3], (4.7), we define

$$\widehat{\alpha}_{j,m,k} = \begin{cases} \widehat{\sigma}_k^{1,1} / \left(\widehat{f}_k^I\right)^2 & \text{for } m = j = k+1, \\ (\widehat{\sigma}_k^{1,2} - \widehat{\sigma}_k^{1,1}) \; / \; (\widehat{f}_k^I \; \widehat{h}_k^I) & \text{for } m > j = k+1 \text{ or } j > m = k+1, \\ \left(\widehat{\sigma}_k^{1,1} - 2\widehat{\sigma}_k^{1,2} + \widehat{\sigma}_k^{2,2}\right) / \left(\widehat{h}_k^I\right)^2 & \text{for } m \geq j > k+1 \text{ or } j \geq m > k+1. \end{cases} \tag{34}$$

Then, from formula (58) in the appendix we obtain the following estimator:

Result 4.2 (Process variance term for a single accident year) *For $i \in \{1, \ldots, I\}$ we obtain the following estimator for the first term on the right-hand side of* (29)

$$\widehat{\mathrm{Var}}\left(\widehat{\mathrm{CDR}}_i(I+1)\Big|\, \mathcal{D}_I\right) = \sum_{j,m=I-i+1}^{I} (\widehat{X}_{i,j}^{Pa})^I (\widehat{X}_{i,m}^{Pa})^I \left\{ \frac{\widehat{\alpha}_{j,m,I-i}}{R_{i,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2 \, \frac{\widehat{\alpha}_{j,m,k}}{R_{I-k,k}} \right\}, \tag{35}$$

where an empty sum is set to zero.

There remains the study of the parameter estimation error term, that is, we would like to know, how much $R_{i,I-i}^2 \Delta_i$ deviates from zero (see Corollary 4.1). We study these deviations with the help of the conditional resampling approach described in Wüthrich-Merz [10], Section 3.2.3. In (73), below, we derive the following result:

Result 4.3 (Parameter Estimation Error for a single accident year) *For accident* *year $i \in \{1, \ldots, I\}$ we obtain the estimator*

$$R^2_{i,I-i}\ \widehat{\Delta}_i = \sum_{j,m=I-i+1}^{J} (\widehat{X}^{Pa}_{i,j})^I (\widehat{X}^{Pa}_{i,m})^I \left[\frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{n=0}^{i-1} R_{n,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2\ \frac{\widehat{\alpha}_{j,m,k}}{\sum_{n=0}^{I-k-1} R_{n,k}} \right], \quad (36)$$

where an empty sum is set to zero.

Results 4.2 and 4.3 imply the following estimator for the conditional MSEP of the observable CDR:

Result 4.4 (Conditional MSEP for single accident years) *For accident year $i \in \{1, \ldots, I\}$ we obtain the following estimator for* (29)

$$\begin{aligned} \widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathcal{D}_I}(0) &= \widehat{\mathrm{Var}}\left(\widehat{\mathrm{CDR}}_i(I+1) \middle| \mathcal{D}_I \right) + R^2_{i,I-i}\ \widehat{\Delta}_i \qquad (37) \\ &= \sum_{j,m=I-i+1}^{J} (\widehat{X}^{Pa}_{i,j})^I (\widehat{X}^{Pa}_{i,m})^I \left[\frac{\delta_{I-i}^{-1}\ \widehat{\alpha}_{j,m,I-i}}{\sum_{n=0}^{i-1} R_{n,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \frac{\delta_k\ \widehat{\alpha}_{j,m,k}}{\sum_{n=0}^{I-k-1} R_{n,k}} \right], \end{aligned}$$

where an empty sum is set to zero.

If we compare the conditional MSEP for the observable CDR to the conditional MSEP of the ultimate claim (Estimator 4.1 in Dahms [3]) we observe that for the payout $X_{i,I-i+1}$ in the next accounting year they coincide and for all the future accounting years they are scaled by δ_k^2. This is analogous to the findings in Bühlmann et al. [2].

Of course exactly the same result holds true on claims incurred basis. All claims paid figures need to be replaced by claims incurred figures and $\widehat{\alpha}_{j,m,k}$ needs to be replaced by

$$\widehat{\beta}_{j,m,k} = \begin{cases} \widehat{\sigma}_k^{2,2} / \left(\widehat{g}_k^I\right)^2 & \text{for } m = j = k+1, \\ (\widehat{\sigma}_k^{2,2} - \widehat{\sigma}_k^{1,2}) / (\widehat{g}_k^I \widehat{h}_k^I) & \text{for } m > j = k+1 \text{ or } j > m = k+1, \\ \left(\widehat{\sigma}_k^{1,1} - 2\widehat{\sigma}_k^{1,2} + \widehat{\sigma}_k^{2,2}\right) / \left(\widehat{h}_k^I\right)^2 & \text{for } m \geq j > k+1 \text{ or } j \geq m > k+1. \end{cases} \quad (38)$$

4.2 Aggregated Accident Years

For aggregated accident years we need to study the conditional MSEP

$$\begin{aligned} \mathrm{msep}_{\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1)|\mathcal{D}_I}(0) &= E\left[\left(\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1) - 0\right)^2 \middle| \mathcal{D}_I\right] \\ &= E\left[\left(\sum_{i=1}^I \widehat{C}_{i,J}^I - \sum_{i=1}^I \widehat{C}_{i,J}^{I+1}\right)^2 \middle| \mathcal{D}_I\right]. \end{aligned} \tag{39}$$

We split the conditional MSEP again into process variance term and parameter estimation error term

$$\mathrm{msep}_{\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1)|\mathcal{D}_I}(0) = \mathrm{Var}\left(\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1)\middle| \mathcal{D}_I\right) + E\left[\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1)\middle| \mathcal{D}_I\right]^2. \tag{40}$$

Hence we again need to study these two terms. For the process variance term we obtain

$$\mathrm{Var}\left(\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1)\middle| \mathcal{D}_I\right) = \sum_{i,l=1}^I \mathrm{Cov}\left(\widehat{\mathrm{CDR}}_i(I+1)\ ,\ \widehat{\mathrm{CDR}}_l(I+1)\middle| \mathcal{D}_I\right). \tag{41}$$

Using the result for single accident years and formula (65) below this allows for the following estimator:

Result 4.5 (Process variance for aggregated accident years) *The process variance term is estimated by*

$$\begin{aligned} \widehat{\mathrm{Var}}\left(\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1)\middle| \mathcal{D}_I\right) &= \sum_{i=1}^I \widehat{\mathrm{Var}}\left(\widehat{\mathrm{CDR}}_i(I+1)\middle| \mathcal{D}_I\right) \\ &+2 \sum_{1\le i<n\le I}\ \sum_{j,m=I-i+1}^{I} (\widehat{X}_{i,j}^{Pa})^I (\widehat{X}_{n,m}^{Pa})^I \left\{ \delta_{I-i}\ \frac{\widehat{\alpha}_{j,m,I-i}}{R_{i,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2\ \frac{\widehat{\alpha}_{j,m,k}}{R_{I-k,k}} \right\}, \end{aligned} \tag{42}$$

where an empty sum is set to zero.

Hence, there remains the study of the parameter estimation error for aggregated accident years which is given by

$$E\left[\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1)\middle| \mathcal{D}_I\right]^2 = \left(\sum_{i=1}^I \widehat{C}_{i,J}^I - \sum_{i=1}^I E\left[\widehat{C}_{i,J}^{I+1}\middle| \mathcal{D}_I\right]\right)^2 \stackrel{def.}{=} \Delta. \tag{43}$$

This term is derived and estimated in (76) which leads to the following estimator ($\Delta_{i,n}$ is defined in (74) below).

Result 4.6 (Parameter Estimation Error for aggregated accident years) *The parameter estimation error term is estimated by*

$$\begin{aligned}\widehat{\Delta} &= \sum_{i=1}^{I} R_{i,I-i}^2 \ \widehat{\Delta}_i + 2 \sum_{1\le i<n\le I} R_{i,I-i}\ R_{n,I-n}\ \widehat{\Delta}_{i,n} \\ &= \sum_{i=1}^{I} R_{i,I-i}^2 \ \widehat{\Delta}_i \qquad (44) \\ &\quad +2 \sum_{1\le i<n\le I} \sum_{j,m=I-i+1}^{J} (\widehat{X}_{i,j}^{Pa})^I (\widehat{X}_{n,m}^{Pa})^I \left[\delta_{I-i} \ \frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2 \ \frac{\widehat{\alpha}_{j,m,k}}{\sum_{l=0}^{I-k-1} R_{l,k}} \right].\end{aligned}$$

Results 4.2-4.6 lead to the following estimator for the conditional MSEP for the observable CDR that is predicted by zero:

Result 4.7 (Conditional MSEP for aggregated accident years) *The MSEP is estimated by*

$$\begin{aligned}&\widehat{\mathrm{msep}}_{\sum_{i=1}^I \widehat{\mathrm{CDR}}_i(I+1)|\mathcal{D}_I}(0) = \sum_{i=1}^{I} \widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathcal{D}_I}(0) \qquad (45) \\ &\quad +2 \sum_{1\le i<n\le I} \sum_{j,m=I-i+1}^{J} (\widehat{X}_{i,j}^{Pa})^I (\widehat{X}_{n,m}^{Pa})^I \left[\frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k \ \frac{\widehat{\alpha}_{j,m,k}}{\sum_{l=0}^{I-k-1} R_{l,k}} \right].\end{aligned}$$

Comparing this to Estimator 4.3 in Dahms [3] we obtain the same picture as for single accident years, namely that for the payouts in the next accounting year we face the full uncertainty whereas for later payouts the uncertainties are scaled by δ_k. For claims incurred figures we need to replace $\widehat{\alpha}_{j,m,k}$ by $\widehat{\beta}_{j,m,k}$.

5 EXAMPLE

We revisit Example 1 given in Dahms [3]. The data is given in Tables 4 and 5 in Dahms [3] and we choose the parameter estimators $\widehat{f}_j^I$, $\widehat{g}_j^I$, $\widehat{h}_j^I$ and $\widehat{\sigma}_j^{m,n}$ provided in Tables 1-3 in Dahms [3] (see also Appendix 11 below). In Figure 2 we give the claims reserves for the outstanding loss liabilities. This is done with the distribution-free chain ladder (CL) method (see Mack [5]) for claims paid (CL Paid) and claims incurred (CL Incurred) data, the complementary loss ratio method (CLRM) and the Munich chain ladder (MCL) method (see Quarg-Mack [9]) for claims paid (MCL Paid) and claims incurred (MCL Incurred) data. We see that the difference between CL Paid reserves and CL Incurred

accident year	reserves CL Paid	reserves CL Incurred	reserves CLRM	reserves MCL Paid	reserves MCL Incurred
1	114'086	337'984	314'902	104'606	338'200
2	394'121	31'884	66'994	457'484	30'850
3	608'749	331'436	359'384	664'871	330'205
4	697'742	1'018'350	981'883	615'436	1'021'361
5	1'234'157	1'103'928	1'115'768	1'271'110	1'102'396
6	1'138'623	1'868'664	1'786'947	919'102	1'894'861
7	1'638'793	1'997'651	1'942'518	1'498'163	2'020'310
8	2'359'939	1'418'779	1'569'657	3'181'319	1'320'492
9	1'979'401	2'556'612	2'590'718	1'602'089	2'703'242
Total	10'165'612	10'665'287	10'728'771	10'314'181	10'761'918

Figure 2: Claims reserves from the distribution-free chain ladder (CL) method, the complementary loss ratio method (CLRM) and the Munich chain ladder (MCL) method for claims paid and claims incurred data.

reserves is about 5%. This gap is slightly smaller if we use the MCL method. The CLRM gives one estimate that is based on both sources of data. In our case the CLRM estimate is higher than both CL estimates but inbetween the two MCL estimates. Probably the payment based reserves underestimate the outstanding loss liabilities. Note also that the fluctuation between different accident years is rather high, especially for payment based methods.

The next Figure 3 provides the conditional MSEP estimator, when the ultimate claim

accident year	$\mathrm{MSEP}^{1/2}$ ultimate $\widehat{C}^I_{i,J}$ CL Paid	$\mathrm{MSEP}^{1/2}$ ultimate $\widehat{C}^I_{i,J}$ CL Incurred	$\mathrm{MSEP}^{1/2}$ ultimate $\widehat{C}^I_{i,J}$ CLRM Paid	$\mathrm{MSEP}^{1/2}$ ultimate $\widehat{C}^I_{i,J}$ CLRM Incurred
1	89'423	2'553	194	14'639
2	234'652	5'186	4'557	5'538
3	255'590	9'264	10'541	12'566
4	261'272	10'874	36'792	38'250
5	323'859	33'243	43'940	44'835
6	274'914	55'884	65'055	65'909
7	373'587	165'086	176'706	176'977
8	492'815	209'162	197'781	197'917
9	468'074	321'560	322'900	323'049
Total	1'517'480	455'794	467'814	471'873

Figure 3: MSEP for the ultimate claim prediction $\widehat{C}^I_{i,J}$ for $C_{i,J}$ from the distribution-free chain ladder (CL) method and the complementary loss ratio method (CLRM) for claims paid and claims incurred data.

$C_{i,J}$ is predicted by $\widehat{C}^I_{i,J}$, i.e.

$$\mathrm{msep}_{C_{i,j}|\mathcal{D}_I}\left(\widehat{C}^I_{i,J}\right) = E\left[\left(C_{i,J} - \widehat{C}^I_{i,J}\right)^2 \middle| \mathcal{D}_I\right]. \tag{46}$$

The estimator in the distribution-free CL method is provided in Mack [5] and for the CLRM in Dahms [3] (for the MCL method there is, so far, no MSEP formula). In contrary to the one-year CDR view, formula (46) is a long term view that quantifies the uncertainty over the whole runoff period. Note that CL Paid MSEP is very high, i.e. the payment based reserves have a high uncertainty. This comes from the rather large volatility in the payment data and also supports our findings that payment based reserves are rather low. The remaining three MSEP estimates are close (as the underlying claims reserves).

Figure 4 provides the conditional MSEP estimates for the CDR predictions. The formula

accident year	$MSEP^{1/2}$ CDR CL Paid	$MSEP^{1/2}$ CDR CL Incurred	$MSEP^{1/2}$ CDR CLRM Paid	$MSEP^{1/2}$ CDR CLRM Incurred
1	89’423	2’553	194	14’639
2	212’824	4’561	4’557	4’678
3	131’568	7’825	5’597	6’628
4	161’173	6’666	33’675	34’258
5	145’918	31’325	30’574	30’997
6	104’760	45’866	42’598	43’074
7	230’692	155’175	166’154	166’255
8	283’635	150’874	138’685	138’740
9	229’060	223’142	210’899	210’979
Total	1’004’164	347’698	346’576	350’534

Figure 4: MSEP for the CDR from the distribution-free chain ladder (CL) method and the complementary loss ratio method (CLRM) for claims paid and claims incurred data.

in the distribution-free CL method is provided in Merz-Wüthrich [7], Result 3.5, and we compare it to the CLRM results provided in Result 4.7 above. The CDR results provide now the one-year solvency view. We see that the one-year CDR picture is similar to the total runoff uncertainty picture. If we choose, for example, the CLRM results they say that it is not unlikely that the claims development result deviates from zero by about 3% of the total reserves. This highlights that the claims reserve development is a substantial source of uncertainty in the earning statements of non-life insurance companies.

Finally, Figure 5 provides the ratios between the one-year CDR uncertainty from Figure 4 and the total runoff uncertainty from Figure 3. We see that the one-year uncertainty is between 60% and 80% of the total runoff uncertainty. These findings are inline with the field study presented in AISAM-ACME [1].

accident year	$\text{MSEP}^{1/2}$ CDR/Ultimate CL Paid	$\text{MSEP}^{1/2}$ CDR/Ultimate CL Incurred	$\text{MSEP}^{1/2}$ CDR/Ultimate CLRM Paid	$\text{MSEP}^{1/2}$ CDR/Ultimate CLRM Incurred
1	100.0%	100.0%	100.0%	100.0%
2	90.7%	87.9%	100.0%	84.5%
3	51.5%	84.5%	53.1%	52.7%
4	61.7%	61.3%	91.5%	89.6%
5	45.1%	94.2%	69.6%	69.1%
6	38.1%	82.1%	65.5%	65.4%
7	61.8%	94.0%	94.0%	93.9%
8	57.6%	72.1%	70.1%	70.1%
9	48.9%	69.4%	65.3%	65.3%
Total	66.2%	76.3%	74.1%	74.3%

Figure 5: MSEP CDR / MSEP Ultimate from the distribution-free chain ladder (CL) method and the complementary loss ratio method (CLRM) for claims paid and claims incurred data.

Conclusion. The complementary loss ratio method provides an algorithm for claims reserving that is simultaneously based on claims paid and claims incurred data. This is similar to the Munich chain ladder method. However, in contrary to the Munich chain ladder method, the complementary loss ratio method also allows for the study of the mean squarre error of prediction which is probability the most popular uncertainty measure in actuarial practice. Dahms [3] has derived the MSEP formula for the total runoff uncertainty of the ultimate claims. In the present paper we derive the MSEP formula for the one-year claims development uncertainty which is an important quantity for solvency purposes.

6 PRELIMINARIES

Lemma 6.1 *Under Model Assumptions 2.1 we have*

$$E\left[\widehat{C}_{i,J}^{I+1}\Big|\mathcal{D}_I\right]=C_{i,I-i}^{Pa}+R_{i,I-i}f_{I-i}$$
$$+R_{i,I-i}\sum_{j=I-i+2}^{J}h_{I-i}\prod_{k=I-i+1}^{j-2}\Big((1-\delta_k)\ \widehat{h}_k^I+\delta_k\ h_k\Big)\Big((1-\delta_{j-1})\ \widehat{f}_{j-1}^I+\delta_{j-1}\ f_{j-1}\Big).$$

Note that if the true parameters were known successive ultimate claims predictions would give a martingale sequence. Crucial for the proof of Lemma 6.1 will be the property that different accident years i are independent. This will also play a crucial role in all further developments below.

Proof of Lemma 6.1. Using (23) and (25) we have

$$E\left[\widehat{C}_{i,J}^{I+1}\Big|\mathcal{D}_I\right]\ =\ E\left[C_{i,I-i+1}^{Pa}+R_{i,I-i+1}\sum_{j=I-i+2}^{J}\prod_{k=I-i+1}^{j-2}\widehat{h}_k^{I+1}\ \widehat{f}_{j-1}^{I+1}\Bigg|\mathcal{D}_I\right] \tag{47}$$

and applying (9) this leads to

$$E\left[\widehat{C}_{i,J}^{I+1}\Big|\mathcal{D}_I\right]=C_{i,I-i}^{Pa}+R_{i,I-i}f_{I-i}+\sum_{j=I-i+2}^{J}E\left[R_{i,I-i+1}\prod_{k=I-i+1}^{j-2}\widehat{h}_k^{I+1}\ \widehat{f}_{j-1}^{I+1}\Bigg|\mathcal{D}_I\right]. \tag{48}$$

Hence we need to analyze the last term on the right-hand side of (48). By means of (33) we decouple the parameter estimators as follows

$$\widehat{f}_j^{I+1}=\frac{\sum_{i=0}^{I-j}X_{i,j+1}^{Pa}}{\sum_{i=0}^{I-j}R_{i,j}}=(1-\delta_j)\ \widehat{f}_j^I+\frac{X_{I-j,j+1}^{Pa}}{\sum_{i=0}^{I-j}R_{i,j}}.$$

Therefore, given $\mathcal{D}_I$, $\widehat{f}_j^{I+1}$ is only random in the element $X_{I-j,j+1}^{Pa}$. Completely analogous

$$\widehat{g}_j^{I+1}=\frac{\sum_{i=0}^{I-j}X_{i,j+1}^{In}}{\sum_{i=0}^{I-j}R_{i,j}}=(1-\delta_j)\ \widehat{g}_j^I+\frac{X_{I-j,j+1}^{In}}{\sum_{i=0}^{I-j}R_{i,j}},$$

and similarly for $\widehat{h}_j^{I+1}=1+\widehat{g}_j^{I+1}-\widehat{f}_j^{I+1}$. Therefore, all terms in the product term on the right-hand side of (48) are independent, conditional on $\mathcal{D}_I$, because the random variables belong to different accident years, given $\mathcal{D}_I$ (see Model Assumptions 2.1), and, therefore, we have

$$E\left[R_{i,I-i+1}\prod_{k=I-i+1}^{j-2}\widehat{h}_k^{I+1}\ \widehat{f}_{j-1}^{I+1}\middle|\mathcal{D}_I\right] = E\left[R_{i,I-i+1}\middle|\mathcal{D}_I\right]\prod_{k=I-i+1}^{j-2}E\left[\widehat{h}_k^{I+1}\middle|\mathcal{D}_I\right]E\left[\widehat{f}_{j-1}^{I+1}\middle|\mathcal{D}_I\right]$$
$$= R_{i,I-i}\ h_{I-i}\prod_{k=I-i+1}^{j-2}E\left[\widehat{h}_k^{I+1}\middle|\mathcal{D}_I\right]E\left[\widehat{f}_{j-1}^{I+1}\middle|\mathcal{D}_I\right]. \qquad (49)$$

Note that

$$E\left[\widehat{f}_j^{I+1}\middle|\mathcal{D}_I\right] = (1-\delta_j)\ \widehat{f}_j^I + \frac{1}{\sum_{i=0}^{I-j}R_{i,j}}E\left[X_{I-j,j+1}^{Pa}\middle|\mathcal{D}_I\right] = (1-\delta_j)\ \widehat{f}_j^I + \delta_j\ f_j, \qquad (50)$$

and completely analogous for $\widehat{g}_j^{I+1}$ and $\widehat{h}_j^{I+1}$. This implies the statement of the lemma.

□

For the decoupling of the covariance terms we use the following lemma.

Lemma 6.2 *Assume that $(Y_i, Z_i)_{i=1,\dots,I}$ is a sequence of independent random vectors. Hence*

$$\mathrm{Cov}\left(\prod_{i=1}^I Y_i, \prod_{i=1}^I Z_i\right) = \prod_{i=1}^I E\left[Y_i\right]E\left[Z_i\right]\ \left\{\prod_{i=1}^I\left(\frac{\mathrm{Cov}\left(Y_i,Z_i\right)}{E\left[Y_i\right]E\left[Z_i\right]}+1\right)-1\right\}$$
$$\approx \prod_{i=1}^I E\left[Y_i\right]E\left[Z_i\right]\ \sum_{i=1}^I\frac{\mathrm{Cov}\left(Y_i,Z_i\right)}{E\left[Y_i\right]E\left[Z_i\right]},$$

for $\frac{\mathrm{Cov}(Y_i,Z_i)}{E[Y_i]E[Z_i]} \ll 1$.

Proof. The proof is a straighforward calculation with covariances and the linear approximation is obtained as described in Merz-Wüthrich [7], formula (A.1).

□

7 PROCESS ERROR FOR SINGLE ACCIDENT YEARS

We do the following abbreviation because it will be constantly used in the sequel

$$\widehat{F}_{i,j-1}^{I+1} = \prod_{k=I-i+1}^{j-2}\widehat{h}_k^{I+1}\ \widehat{f}_{j-1}^{I+1}. \qquad (51)$$

This is the factor that is needed to get the predictor $(\widehat{X}_{i,j}^{Pa})^{I+1}$ at time $I+1$ (cf. (23)).

In this appendix we derive an estimator for the process error term in (29). Note that due to the $\mathcal{D}_I$-measurability of $\widehat{C}_{i,J}^{I}$ we have

$$\begin{aligned}
\mathrm{Var}\left(\widehat{\mathrm{CDR}}_i(I+1)\middle|\mathcal{D}_I\right) &= \mathrm{Var}\left(\widehat{C}_{i,J}^{I+1}\middle|\mathcal{D}_I\right) \\
&= \mathrm{Var}\left(C_{i,I-i+1}^{Pa} + R_{i,I-i+1}\sum_{j=I-i+2}^{J}\widehat{F}_{i,j-1}^{I+1}\middle|\mathcal{D}_I\right) \\
&= \mathrm{Var}\left(C_{i,I-i+1}^{Pa}\middle|\mathcal{D}_I\right) + \mathrm{Var}\left(R_{i,I-i+1}\sum_{j=I-i+2}^{J}\widehat{F}_{i,j-1}^{I+1}\middle|\mathcal{D}_I\right) \\
&\quad + 2\sum_{j=I-i+2}^{J}\mathrm{Cov}\left(C_{i,I-i+1}^{Pa}, R_{i,I-i+1}\,\widehat{F}_{i,j-1}^{I+1}\middle|\mathcal{D}_I\right).
\end{aligned} \tag{52}$$

We derive estimates for all the terms on the right-hand side of (52). For the first term we obtain

$$\mathrm{Var}\left(C_{i,I-i+1}^{Pa}\middle|\mathcal{D}_I\right) = \mathrm{Var}\left(X_{i,I-i+1}^{Pa}\middle|\mathcal{D}_I\right) = R_{i,I-i}\;\sigma_{I-i}^{1,1}.$$

An estimator is obtained by replacing the parameter $\sigma_{I-i}^{1,1}$ by its estimator at time I, i.e.

$$\widehat{\mathrm{Var}}\left(C_{i,I-i+1}^{Pa}\middle|\mathcal{D}_I\right) = R_{i,I-i}\;\widehat{\sigma}_{I-i}^{1,1} = \left[(\widehat{X}_{i,I-i+1}^{Pa})^{I}\right]^2 \frac{\widehat{\alpha}_{I-i+1,I-i+1,I-i}}{R_{i,I-i}}. \tag{53}$$

For the third term on the right-hand side of (52) we have (similar as in the proof of Lemma 6.1 we use the independence of different accident years)

$$\begin{aligned}
&\sum_{j=I-i+2}^{J}\mathrm{Cov}\left(C_{i,I-i+1}^{Pa}, R_{i,I-i+1}\;\widehat{F}_{i,j-1}^{I+1}\middle|\mathcal{D}_I\right) \\
&\quad= \sum_{j=I-i+2}^{J}\mathrm{Cov}\left(X_{i,I-i+1}^{Pa}, R_{i,I-i+1}\middle|\mathcal{D}_I\right)\prod_{k=I-i+1}^{j-2} E\left[\widehat{h}_k^{I+1}\middle|\mathcal{D}_I\right]\; E\left[\widehat{f}_{j-1}^{I+1}\middle|\mathcal{D}_I\right] \\
&\quad= \sum_{j=I-i+2}^{J} R_{i,I-i}\left(\sigma_{I-i}^{1,2}-\sigma_{I-i}^{1,1}\right)\prod_{k=I-i+1}^{j-2} E\left[\widehat{h}_k^{I+1}\middle|\mathcal{D}_I\right]\; E\left[\widehat{f}_{j-1}^{I+1}\middle|\mathcal{D}_I\right].
\end{aligned}$$

If we use now (50) and replace all the parameters by its estimators at time I we obtain the following estimator

$$\sum_{j=I-i+2}^{J} \widehat{\text{Cov}}\left(C^{Pa}_{i,I-i+1}, R_{i,I-i+1}\ \widehat{F}^{I+1}_{i,j-1} \middle| \mathcal{D}_I \right) = R_{i,I-i}\left(\widehat{\sigma}^{1,2}_{I-i} - \widehat{\sigma}^{1,1}_{I-i}\right) \sum_{j=I-i+2}^{J} \prod_{k=I-i+1}^{j-2} \widehat{h}^I_k\ \widehat{f}^I_{j-1}$$

$$= (\widehat{X}^{Pa}_{i,I-i+1})^I \sum_{j=I-i+2}^{J} \frac{\widehat{\alpha}_{I-i+1,j,I-i}}{R_{i,I-i}} (\widehat{X}^{Pa}_{i,j})^I. \tag{54}$$

So there remains the treatment of the middle term on the right-hand side of (52). We get

$$\text{Var}\left(R_{i,I-i+1} \sum_{j=I-i+2}^{J} \widehat{F}^{I+1}_{i,j-1} \middle| \mathcal{D}_I \right) = \sum_{j,m=I-i+2}^{J} \text{Cov}\left(R_{i,I-i+1}\widehat{F}^{I+1}_{i,j-1}\ ,\ R_{i,I-i+1}\widehat{F}^{I+1}_{i,m-1} \middle| \mathcal{D}_I \right). \tag{55}$$

We start with $j = m$. Then the last term in (55) is approximated with the help of Lemma 6.2 (note that $\widehat{F}^{I+1}_{j-1}$ is a product of independent random variables, given $\mathcal{D}_I$) by

$$\text{Var}\left(R_{i,I-i+1}\widehat{F}^{I+1}_{i,j-1} \middle| \mathcal{D}_I \right) \approx E\left[R_{i,I-i+1} \middle| \mathcal{D}_I \right]^2 \prod_{k=I-i+1}^{j-2} E\left[\widehat{h}^{I+1}_k \middle| \mathcal{D}_I \right]^2 E\left[\widehat{f}^{I+1}_{j-1} \middle| \mathcal{D}_I \right]^2$$

$$\times \left\{ \frac{\text{Var}\left(R_{i,I-i+1} \middle| \mathcal{D}_I \right)}{E\left[R_{i,I-i+1} \middle| \mathcal{D}_I \right]^2} + \sum_{k=I-i+1}^{j-2} \frac{\text{Var}\left(\widehat{h}^{I+1}_k \middle| \mathcal{D}_I \right)}{E\left[\widehat{h}^{I+1}_k \middle| \mathcal{D}_I \right]^2} + \frac{\text{Var}\left(\widehat{f}^{I+1}_{j-1} \middle| \mathcal{D}_I \right)}{E\left[\widehat{f}^{I+1}_{j-1} \middle| \mathcal{D}_I \right]^2} \right\}. \tag{56}$$

The first term is treated as in (49)-(50) and if we replace all parameters by their estimators we obtain

$$\widehat{E}\left[R_{i,I-i+1} \middle| \mathcal{D}_I \right]^2 \prod_{k=I-i+1}^{j-2} \widehat{E}\left[\widehat{h}^{I+1}_k \middle| \mathcal{D}_I \right]^2 \widehat{E}\left[\widehat{f}^{I+1}_{j-1} \middle| \mathcal{D}_I \right]^2$$

$$= \widehat{E}\left[R_{i,I-i+1} \prod_{k=I-i+1}^{j-2} \widehat{h}^{I+1}_k\ \widehat{f}^{I+1}_{j-1} \middle| \mathcal{D}_I \right]^2 = \left(R_{i,I-i} \prod_{k=I-i}^{j-2} \widehat{h}^I_k\ \widehat{f}^I_{j-1} \right)^2 = \left[(\widehat{X}^{Pa}_{i,j})^I \right]^2.$$

Moreover we obtain the following estimators

$$\widehat{\text{Var}}\left(R_{i,I-i+1} \middle| \mathcal{D}_I \right) = R_{i,I-i}\left(\widehat{\sigma}^{1,1}_{I-i} - 2\widehat{\sigma}^{1,2}_{I-i} + \widehat{\sigma}^{2,2}_{I-i} \right).$$

The last term on the right-hand side of (56) is estimated by

$$\begin{aligned} \widehat{\text{Var}}\left(\widehat{f}^{I+1}_{j-1} \middle| \mathcal{D}_I \right) &= \left(\frac{1}{\sum_{i=0}^{I-j+1} R_{i,j-1}} \right)^2 \widehat{\text{Var}}\left(X^{Pa}_{I-j+1,j} \middle| \mathcal{D}_I \right) \\ &= \frac{R_{I-j+1,j-1}}{\left(\sum_{i=0}^{I-j+1} R_{i,j-1} \right)^2}\ \widehat{\sigma}^{1,1}_{j-1} = \delta^2_{j-1}\ \frac{\widehat{\sigma}^{1,1}_{j-1}}{R_{I-j+1,j-1}}, \end{aligned}$$

and analogously

$$\widehat{\mathrm{Var}}\left(\widehat{h}_k^{I+1}\middle|\mathcal{D}_I\right)=\delta_k^2\;\frac{\widehat{\sigma}_k^{1,1}-2\widehat{\sigma}_k^{1,2}+\widehat{\sigma}_k^{2,2}}{R_{I-k,k}}.$$

This gives for $j=m$ the estimator

$$\widehat{\mathrm{Var}}\left(R_{i,I-i+1}\widehat{F}_{i,j-1}^{I+1}\middle|\mathcal{D}_I\right)=\left[(\widehat{X}_{i,j}^{Pa})^I\right]^2\left\{\frac{\widehat{\alpha}_{j,j,I-i}}{R_{i,I-i}}+\sum_{k=I-i+1}^{j-1}\delta_k^2\;\frac{\widehat{\alpha}_{j,j,k}}{R_{I-k,k}}\right\}.$$

And in complete analogy we obtain for $m>j$ or $j>m$

$$\widehat{\mathrm{Cov}}\left(R_{i,I-i+1}\widehat{F}_{i,j-1}^{I+1},R_{i,I-i+1}\widehat{F}_{i,m-1}^{I+1}\middle|\mathcal{D}_I\right)=(\widehat{X}_{i,j}^{Pa})^I(\widehat{X}_{i,m}^{Pa})^I\left\{\frac{\widehat{\alpha}_{j,m,I-i}}{R_{i,I-i}}+\sum_{k=I-i+1}^{\min\{j,m\}-1}\delta_k^2\frac{\widehat{\alpha}_{j,m,k}}{R_{I-k,k}}\right\}. \tag{57}$$

Collecting all the terms we obtain for (52) the following estimator

$$\begin{aligned}\widehat{\mathrm{Var}}\left(\widehat{\mathrm{CDR}}_i(I+1)\middle|\mathcal{D}_I\right)\;&=\;\widehat{\mathrm{Var}}\left(\widehat{C}_{i,J}^{I+1}\middle|\mathcal{D}_I\right)\\ =\;&\left[(\widehat{X}_{i,I-i+1}^{Pa})^I\right]^2\;\frac{\widehat{\alpha}_{I-i+1,I-i+1,I-i}}{R_{i,I-i}}\\ &+\sum_{j,m=I-i+2}^{I}(\widehat{X}_{i,j}^{Pa})^I(\widehat{X}_{i,m}^{Pa})^I\left\{\frac{\widehat{\alpha}_{j,m,I-i}}{R_{i,I-i}}+\sum_{k=I-i+1}^{\min\{j,m\}-1}\delta_k^2\;\frac{\widehat{\alpha}_{j,m,k}}{R_{I-k,k}}\right\}\\ &+2\;(\widehat{X}_{i,I-i+1}^{Pa})^I\sum_{j=I-i+2}^{J}\frac{\widehat{\alpha}_{I-i+1,j,I-i}}{R_{i,I-i}}(\widehat{X}_{i,j}^{Pa})^I.\end{aligned} \tag{58}$$

But from this the estimator (35) follows. This completes the derivation of Result 4.2.

8 PROCESS ERROR FOR AGGREGATE ACCIDENT YEARS

In this appendix we derive an estimator for the process error term given in (41). We choose $i<n$ Note that due to the $\mathcal{D}_I$-measurability of $\widehat{C}_{i,J}^I$ and $\widehat{C}_{n,J}^I$ we have

$$\mathrm{Cov}\left(\widehat{\mathrm{CDR}}_i(I+1),\widehat{\mathrm{CDR}}_n(I+1)\middle|\mathcal{D}_I\right)=\mathrm{Cov}\left(\widehat{C}_{i,J}^{I+1},\widehat{C}_{n,J}^{I+1}\middle|\mathcal{D}_I\right).$$

Hence we need to calculate the covariance between ultimate prediction for different accident years $i < n$.

$$\begin{aligned}\mathrm{Cov}\left(\widehat{C}^{I+1}_{i,J}, \widehat{C}^{I+1}_{n,J}\Big|\, \mathcal{D}_I\right) &= \mathrm{Cov}\left(C^{Pa}_{i,I-i+1}, C^{Pa}_{n,I-n+1}\big|\, \mathcal{D}_I\right) \\ &+ \mathrm{Cov}\left(R_{i,I-i+1} \sum_{j=I-i+2}^{J} \widehat{F}^{I+1}_{i,j-1}\ ,\ R_{n,I-n+1} \sum_{m=I-n+2}^{J} \widehat{F}^{I+1}_{n,m-1}\Bigg|\, \mathcal{D}_I\right) \\ &+ \mathrm{Cov}\left(C^{Pa}_{i,I-i+1}, R_{n,I-n+1} \sum_{m=I-n+2}^{J} \widehat{F}^{I+1}_{n,m-1}\Bigg|\, \mathcal{D}_I\right) \\ &+ \mathrm{Cov}\left(C^{Pa}_{n,I-n+1}, R_{i,I-i+1} \sum_{j=I-i+2}^{J} \widehat{F}^{I+1}_{i,j-1}\Bigg|\, \mathcal{D}_I\right).\end{aligned} \tag{59}$$

We derive estimates for all the terms on the right-hand side of (59). For the first term we obtain

$$\mathrm{Cov}\left(C^{Pa}_{i,I-i+1}, C^{Pa}_{n,I-n+1}\big|\, \mathcal{D}_I\right) = \mathrm{Cov}\left(X^{Pa}_{i,I-i+1}, X^{Pa}_{n,I-n+1}\big|\, \mathcal{D}_I\right) = 0, \tag{60}$$

because the two random variables correspond to different accident years. Analogously, we have for $i < n$

$$\mathrm{Cov}\left(C^{Pa}_{n,I-n+1}, R_{i,I-i+1} \sum_{j=I-i+2}^{J} \widehat{F}^{I+1}_{i,j-1}\Bigg|\, \mathcal{D}_I\right) = 0. \tag{61}$$

Therefore, only the two middle terms on the right-hand side of (59) need a treatment. We start with the third term. Note that $i < n$ implies $I - i + 1 > I - n + 1$. Using the independence between different accident years we get

$$\begin{aligned}&\sum_{m=I-n+2}^{J} \mathrm{Cov}\left(C^{Pa}_{i,I-i+1}, R_{n,I-n+1}\ \widehat{F}^{I+1}_{n,m-1}\Big|\, \mathcal{D}_I\right) \\ &\quad= \sum_{m=I-i+2}^{J} E\left[R_{n,I-n+1}\big|\, \mathcal{D}_I\right] \prod_{k=I-n+1}^{I-i-1} E\left[\widehat{h}^{I+1}_{k}\Big|\, \mathcal{D}_I\right] \mathrm{Cov}\left(C^{Pa}_{i,I-i+1}, \widehat{h}^{I+1}_{I-i}\Big|\, \mathcal{D}_I\right) \\ &\qquad\qquad \times \prod_{k=I-i+1}^{m-2} E\left[\widehat{h}^{I+1}_{k}\Big|\, \mathcal{D}_I\right]\ E\left[\widehat{f}^{I+1}_{m-1}\Big|\, \mathcal{D}_I\right] \\ &\qquad + E\left[R_{n,I-n+1}\big|\, \mathcal{D}_I\right] \prod_{k=I-n+1}^{I-i-1} E\left[\widehat{h}^{I+1}_{k}\Big|\, \mathcal{D}_I\right] \mathrm{Cov}\left(C^{Pa}_{i,I-i+1}, \widehat{f}^{I+1}_{I-i}\Big|\, \mathcal{D}_I\right).\end{aligned}$$

If we calculate all the terms and replace the parameters by its estimators at time I we obtain

$$\sum_{m=I-n+2}^{J} \widehat{\mathrm{Cov}}\left(C^{Pa}_{i,I-i+1}, R_{n,I-n+1}\ \widehat{F}^{I+1}_{n,m-1}\middle|\,\mathcal{D}_I\right) \tag{62}$$

$$= \sum_{m=I-i+2}^{J} R_{n,I-n} \prod_{k=I-n}^{I-i-1} \widehat{h}^I_k\ \delta_{I-i}\left(\widehat{\sigma}^{1,2}_{I-i} - \widehat{\sigma}^{1,1}_{I-i}\right) \prod_{k=I-i+1}^{m-2} \widehat{h}^I_k \widehat{f}^I_{m-1} + R_{n,I-n} \prod_{k=I-n}^{I-i-1} \widehat{h}^I_k\ \delta_{I-i}\widehat{\sigma}^{1,1}_{I-i}$$

$$= (\widehat{X}^{Pa}_{i,I-i+1})^I \sum_{m=I-i+1}^{J} \delta_{I-i} \frac{\widehat{\alpha}_{I-i+1,m,I-i}}{R_{i,I-i}} (\widehat{X}^{Pa}_{n,m})^I.$$

It remains the second term on the right-hand side of (59) to be considered for $i < n$. Note that if $m-1 < I-i$ then only different accident years are considered in the sums in the next displayed formulas. Hence we obtain

$$\mathrm{Cov}\left(R_{i,I-i+1} \sum_{j=I-i+2}^{J} \widehat{F}^{I+1}_{i,j-1}, R_{n,I-n+1} \sum_{m=I-n+2}^{J} \widehat{F}^{I+1}_{n,m-1}\middle|\,\mathcal{D}_I\right)$$

$$= \sum_{j,m=I-i+2}^{J} \mathrm{Cov}\left(R_{i,I-i+1}\widehat{F}^{I+1}_{i,j-1}, R_{n,I-n+1}\widehat{F}^{I+1}_{n,m-1}\middle|\,\mathcal{D}_I\right) \tag{63}$$

$$+ \sum_{j=I-i+2}^{J} \mathrm{Cov}\left(R_{i,I-i+1}\widehat{F}^{I+1}_{i,j-1}, R_{n,I-n+1}\widehat{F}^{I+1}_{n,I-i}\middle|\,\mathcal{D}_I\right).$$

The second term on the right-hand side of (63) is straightforward and is estimated as above by

$$\sum_{j=I-i+2}^{J} \widehat{\mathrm{Cov}}\left(R_{i,I-i+1}\widehat{F}^{I+1}_{i,j-1}, R_{n,I-n+1}\widehat{F}^{I+1}_{n,I-i}\middle|\,\mathcal{D}_I\right)$$

$$= \sum_{j=I-i+2}^{J} (\widehat{X}^{Pa}_{i,j})^I\ \delta_{I-i}\frac{\widehat{\alpha}_{j,I-i+1,I-i}}{R_{i,I-i}}\ (\widehat{X}^{Pa}_{n,I-i+1})^I.$$

The first term on the right-hand side of (63) needs more care. We have

$$\sum_{j,m=I-i+2}^{J} \mathrm{Cov}\left(R_{i,I-i+1}\widehat{F}^{I+1}_{i,j-1}, R_{n,I-n+1}\widehat{F}^{I+1}_{n,m-1}\middle|\,\mathcal{D}_I\right)$$

$$= \sum_{j,m=I-i+2}^{J} \mathrm{Cov}\left(R_{i,I-i+1}\widehat{F}^{I+1}_{i,j-1}, \widehat{h}^{I+1}_{I-i}\widehat{F}^{I+1}_{i,m-1}\middle|\,\mathcal{D}_I\right) E\left[R_{n,I-n+1}\middle|\,\mathcal{D}_I\right] \prod_{k=I-n+1}^{I-i-1} E\left[\widehat{h}^{I+1}_k\middle|\,\mathcal{D}_I\right].$$

Note that it was our intention to decouple the product in the second argument of the covariance function. This should now be compared to (55). We see that $R_{i,I-i+1} = R_{i,I-i} + X^{In}_{i,I-i+1} - X^{Pa}_{i,I-i+1}$ in the second argument of the covariance function is now replaced by

$$\begin{aligned}\widehat{h}^{I+1}_{I-i} &= 1 + (1-\delta_{I-i})\ \left(\widehat{g}^I_{I-i} - \widehat{f}^I_{I-i}\right) + \frac{X^{In}_{i,I-i+1} - X^{Pa}_{i,I-i+1}}{\sum_{l=0}^{i} R_{l,I-i}} \\ &= 1 + (1-\delta_{I-i})\ \left(\widehat{g}^I_{I-i} - \widehat{f}^I_{I-i}\right) + \delta_{I-i}\frac{X^{In}_{i,I-i+1} - X^{Pa}_{i,I-i+1}}{R_{i,I-i}}.\end{aligned}$$

Hence the stochastic terms only differ in the factor $\delta_{I-i}/R_{i,I-i}$, given $\mathcal{D}_I$. Therefore we may apply the same results as for (55) and obtain the following estimator

$$\begin{aligned}&\sum_{j,m=I-i+2}^{J} \widehat{\mathrm{Cov}}\left(R_{i,I-i+1}\widehat{F}^{I+1}_{i,j-1}, R_{n,I-n+1}\widehat{F}^{I+1}_{n,m-1}\middle|\mathcal{D}_I\right) \qquad (64)\\ &\quad= \sum_{j,m=I-i+2}^{I} (\widehat{X}^{Pa}_{i,j})^I(\widehat{X}^{Pa}_{n,m})^I \left\{\delta_{I-i}\ \frac{\widehat{\alpha}_{j,m,I-i}}{R_{i,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta^2_k\ \frac{\widehat{\alpha}_{j,m,k}}{R_{I-k,k}}\right\}.\end{aligned}$$

Hence we obtain the following estimator for the covariance terms for $i < n$

$$\begin{aligned}&\widehat{\mathrm{Cov}}\left(\widehat{\mathrm{CDR}}_i(I+1), \widehat{\mathrm{CDR}}_n(I+1)\middle|\mathcal{D}_I\right) \qquad (65)\\ &\quad= \sum_{j,m=I-i+2}^{I} (\widehat{X}^{Pa}_{i,j})^I(\widehat{X}^{Pa}_{n,m})^I \left\{\delta_{I-i}\ \frac{\widehat{\alpha}_{j,m,I-i}}{R_{i,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta^2_k\ \frac{\widehat{\alpha}_{j,m,k}}{R_{I-k,k}}\right\} \\ &\qquad + \sum_{j=I-i+2}^{J} (\widehat{X}^{Pa}_{i,j})^I\ \delta_{I-i}\frac{\widehat{\alpha}_{j,I-i+1,I-i}}{R_{i,I-i}}\ (\widehat{X}^{Pa}_{n,I-i+1})^I \\ &\qquad + (\widehat{X}^{Pa}_{i,I-i+1})^I \sum_{j=I-i+1}^{J} \delta_{I-i}\frac{\widehat{\alpha}_{I-i+1,j,I-i}}{R_{i,I-i}}(\widehat{X}^{Pa}_{n,j})^I \\ &\quad= \sum_{j,m=I-i+1}^{I} (\widehat{X}^{Pa}_{i,j})^I(\widehat{X}^{Pa}_{n,m})^I \left\{\delta_{I-i}\ \frac{\widehat{\alpha}_{j,m,I-i}}{R_{i,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta^2_k\ \frac{\widehat{\alpha}_{j,m,k}}{R_{I-k,k}}\right\}.\end{aligned}$$

This and Result 4.2 provide Result 4.5.

9 PARAMETER ESTIMATION ERROR, SINGLE ACCIDENT YEARS

We study the parameter estimation error term $R^2_{i,I-i}\ \Delta_i$ with the help of the conditional resampling approach described in Wüthrich-Merz [10], Section 3.2.3. In the conditional resampling approach one resamples always only the next observation in the time series using the preceding observation as a deterministic volume measure. From Corollary 4.1 we see that we need to study

$$R^2_{i,I-i}\ \Delta_i \quad = \quad R^2_{i,I-i} \left\{ \left(\widehat{f}^I_{I-i} - f_{I-i} \right) + \sum_{j=I-i+2}^{J} Z_{I-i,j-1} \right\}^2, \tag{66}$$

with

$$Z_{I-i,j-1} = \prod_{k=I-i}^{j-2} \widehat{h}^I_k\, \widehat{f}^I_{j-1} - h_{I-i} \prod_{k=I-i+1}^{j-2} \Big((1-\delta_k)\ \widehat{h}^I_k + \delta_k\ h_k \Big) \Big((1-\delta_{j-1})\ \widehat{f}^I_{j-1} + \delta_{j-1}\ f_{j-1} \Big).$$

We would like to apply the conditional resampling approach to the term in the brackets in (66), i.e. Δ_i. As in Wüthrich-Merz [10] we denote the measure for the conditional resampling by $P^*_{\mathcal{D}_I}$, the expectation, variance and covariance w.r.t. $P^*_{\mathcal{D}_I}$ by $E^*_{\mathcal{D}_I}$, $\mathrm{Var}^*_{\mathcal{D}_I}$ and $\mathrm{Cov}^*_{\mathcal{D}_I}$, respectively. For a detailed description of the choice of the conditional resampling measure we refer to Section 3.2.3 in Wüthrich-Merz [10]. Under the conditional resampling measure $P^*_{\mathcal{D}_I}$ we have the following properties (this is all that is needed for our derivations):

(1) any arbitrary collection of resampled estimates $\widehat{f}^I_j$ and $\widehat{g}^I_j$ with different indices are independent under $P^*_{\mathcal{D}_I}$.

(2) $E^*_{\mathcal{D}_I}\left[\widehat{f}^I_j\right] = f_j$ and $E^*_{\mathcal{D}_I}\left[\widehat{g}^I_j\right] = g_j$ for all j,

(3) $\mathrm{Var}^*_{\mathcal{D}_I}\left(\widehat{f}^I_j\right) = \sigma^{1,1}_j / \sum_{i=0}^{I-j-1} R_{i,j}$ and $\mathrm{Var}^*_{\mathcal{D}_I}\left(\widehat{g}^I_j\right) = \sigma^{2,2}_j / \sum_{i=0}^{I-j-1} R_{i,j}$ for all j,

(4) $\mathrm{Cov}^*_{\mathcal{D}_I}\left(\widehat{f}^I_j, \widehat{g}^I_j\right) = \sigma^{1,2}_j / \sum_{i=0}^{I-j-1} R_{i,j}$ for all j.

(5) the weights δ_j, $j = 0, \ldots, J-1$, are constants under $P^*_{\mathcal{D}_I}$.

These properties imply that $E^*_{\mathcal{D}_I}\left[\Delta_i^{1/2}\right]=0$. Therefore, the term in (66) is estimated by $R^2_{i,I-i}E^*_{\mathcal{D}_I}\left[\Delta_i\right]=R^2_{i,I-i}\mathrm{Var}^*_{\mathcal{D}_I}\left(\Delta_i^{1/2}\right)$ under the conditional resampling approach. Therefore we get the estimation

$$R^2_{i,I-i}\mathrm{Var}^*_{\mathcal{D}_I}\left(\Delta_i^{1/2}\right)=R^2_{i,I-i}\mathrm{Var}^*_{\mathcal{D}_I}\left(\left(\widehat{f}^I_{I-i}-f_{I-i}\right)+\sum_{j=I-i+2}^{J}Z_{I-i,j-1}\right). \tag{67}$$

The variance term in (67) is equal to

$$\begin{aligned}\mathrm{Var}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}+\sum_{j=I-i+2}^{J}Z_{I-i,j-1}\right)&=\mathrm{Var}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}\right)\\&+2\sum_{j=I-i+2}^{J}\mathrm{Cov}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i},\widehat{h}^I_{I-i}\right)\prod_{k=I-i+1}^{j-2}h_k\ f_{j-1}+\mathrm{Var}^*_{\mathcal{D}_I}\left(\sum_{j=I-i+2}^{J}Z_{I-i,j-1}\right).\end{aligned} \tag{68}$$

The first two terms on the right-hand side of (68) are straightforward

$$\mathrm{Var}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}\right)=\sigma^{1,1}_{I-i}\Big/\sum_{n=0}^{i-1}R_{n,I-i}, \tag{69}$$

and

$$\sum_{j=I-i+2}^{J}\mathrm{Cov}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i},\widehat{h}^I_{I-i}\right)\prod_{k=I-i+1}^{j-2}h_k\ f_{j-1}=\sum_{j=I-i+2}^{J}\frac{\sigma^{1,2}_{I-i}-\sigma^{1,1}_{I-i}}{\sum_{n=0}^{i-1}R_{n,I-i}}\prod_{k=I-i+1}^{j-2}h_k\ f_{j-1}. \tag{70}$$

The last term on the right-hand side of (68) requires more work. If we extract the sum out of the variance operator we see that we need to study the covariances under the sum

$$\mathrm{Var}^*_{\mathcal{D}_I}\left\{\sum_{j=I-i+2}^{J}Z_{I-i,j-1}\right\}=\sum_{j,m=I-i+2}^{J}\mathrm{Cov}^*_{\mathcal{D}_I}\left\{Z_{I-i,j-1},Z_{I-i,m-1}\right\}.$$

We first study this covariance terms for $j=m$. Then we obtain

$$\begin{aligned}\mathrm{Var}^*_{\mathcal{D}_I}\left\{Z_{I-i,j-1}\right\}&=\mathrm{Var}^*_{\mathcal{D}_I}\left\{\prod_{k=I-i}^{j-2}\widehat{h}^I_k\ \widehat{f}^I_{j-1}\right\}\\&-2h^2_{I-i}\mathrm{Cov}^*_{\mathcal{D}_I}\left\{\prod_{k=I-i+1}^{j-2}\widehat{h}^I_k\ \widehat{f}^I_{j-1},\right.\\&\qquad\left.\prod_{k=I-i+1}^{j-2}\left((1-\delta_k)\ \widehat{h}^I_k+\delta_k\ h_k\right)\left((1-\delta_{j-1})\ \widehat{f}^I_{j-1}+\delta_{j-1}\ f_{j-1}\right)\right\}\\&+h^2_{I-i}\mathrm{Var}^*_{\mathcal{D}_I}\left\{\prod_{k=I-i+1}^{j-2}\left((1-\delta_k)\ \widehat{h}^I_k+\delta_k\ h_k\right)\left((1-\delta_{j-1})\ \widehat{f}^I_{j-1}+\delta_{j-1}\ f_{j-1}\right)\right\}.\end{aligned}$$

To decouple these covariance terms we use Lemma 6.2. The first term is therefore approximated by

$$\begin{aligned}
\mathrm{Var}^*_{\mathcal{D}_I}\left\{\prod_{k=I-i}^{j-2} \widehat{h}_k^I \ \widehat{f}_{j-1}^I\right\} &\approx \prod_{k=I-i}^{j-2} h_k^2 \ f_{j-1}^2 \left[\sum_{k=I-i}^{j-2} \frac{\mathrm{Var}^*_{\mathcal{D}_I}\left(\widehat{h}_k^I\right)}{h_k^2} + \frac{\mathrm{Var}^*_{\mathcal{D}_I}\left(\widehat{f}_{j-1}^I\right)}{f_{j-1}^2}\right] \\
&= \prod_{k=I-i}^{j-2} h_k^2 \ f_{j-1}^2 \left[\sum_{k=I-i}^{j-2} \frac{(\sigma_k^{1,1} - 2\sigma_k^{1,2} + \sigma_k^{2,2})/h_k^2}{\sum_{l=0}^{I-k-1} R_{l,k}} + \frac{\sigma_{j-1}^{1,1}/f_{j-1}^2}{\sum_{l=0}^{I-j} R_{l,j-1}}\right].
\end{aligned}$$

Furthermore, for the second term we obtain

$$\begin{aligned}
&\mathrm{Cov}^*_{\mathcal{D}_I}\left\{\prod_{k=I-i+1}^{j-2} \widehat{h}_k^I \ \widehat{f}_{j-1}^I, \prod_{k=I-i+1}^{j-2} \Big((1-\delta_k)\ \widehat{h}_k^I + \delta_k \ h_k\Big)\Big((1-\delta_{j-1})\ \widehat{f}_{j-1}^I + \delta_{j-1}\ f_{j-1}\Big)\right\} \\
&\approx \prod_{k=I-i+1}^{j-2} h_k^2 \ f_{j-1}^2 \left[\sum_{k=I-i+1}^{j-2} (1-\delta_k)\,\frac{(\sigma_k^{1,1} - 2\sigma_k^{1,2} + \sigma_k^{2,2})/h_k^2}{\sum_{l=0}^{I-k-1} R_{l,k}} + (1-\delta_{j-1})\,\frac{\sigma_{j-1}^{1,1}/f_{j-1}^2}{\sum_{l=0}^{I-j} R_{l,j-1}}\right],
\end{aligned}$$

and for the last term Lemma 6.2 leads to

$$\begin{aligned}
&\mathrm{Var}^*_{\mathcal{D}_I}\left\{\prod_{k=I-i+1}^{j-2} \Big((1-\delta_k)\ \widehat{h}_k^I + \delta_k \ h_k\Big)\Big((1-\delta_{j-1})\ \widehat{f}_{j-1}^I + \delta_{j-1}\ f_{j-1}\Big)\right\} \\
&\approx \prod_{k=I-i+1}^{j-2} h_k^2 \ f_{j-1}^2 \left[\sum_{k=I-i+1}^{j-2} (1-\delta_k)^2\,\frac{(\sigma_k^{1,1} - 2\sigma_k^{1,2} + \sigma_k^{2,2})/h_k^2}{\sum_{l=0}^{I-k-1} R_{l,k}} + (1-\delta_{j-1})^2\,\frac{\sigma_{j-1}^{1,1}/f_{j-1}^2}{\sum_{l=0}^{I-j} R_{l,j-1}}\right].
\end{aligned}$$

Hence this provides for $j = m$ the following estimator, if we replace all parameters by their estimators at time I

$$\left[\prod_{k=I-i}^{j-2} \widehat{h}_k^I \ \widehat{f}_{j-1}^I\right]^2 \left[\frac{\widehat{\alpha}_{j,j,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{j-1} \delta_k^2 \ \frac{\widehat{\alpha}_{j,j,k}}{\sum_{l=0}^{I-k-1} R_{l,k}}\right], \tag{71}$$

and for $j > m$ or $m > j$ we obtain in the same manner the estimator

$$\prod_{k=I-i}^{j-2} \widehat{h}_k^I \ \widehat{f}_{j-1}^I \prod_{k=I-i}^{m-2} \widehat{h}_k^I \ \widehat{f}_{m-1}^I \left[\frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2 \ \frac{\widehat{\alpha}_{j,m,k}}{\sum_{l=0}^{I-k-1} R_{l,k}}\right]. \tag{72}$$

Collecting all the terms and replacing all parameters by their estimators we obtain the following estimator for (66)

$$
\begin{aligned}
& R_{i,I-i}^2 \ \widehat{\Delta}_i \\
= \ & R_{i,I-i}^2 \Bigg\{ \left(\widehat{f}_{I-i}^I\right)^2 \frac{\widehat{\alpha}_{I-i+1,I-i+1,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + 2 \sum_{j=I-i+2}^{J} \widehat{f}_{I-i}^I \prod_{k=I-i}^{j-2} \widehat{h}_k^I \ \widehat{f}_{j-1}^I \frac{\widehat{\alpha}_{I-i+1,j,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} \\
+ \ & \sum_{j,m=I-i+2}^{J} \prod_{k=I-i}^{j-2} \widehat{h}_k^I \ \widehat{f}_{j-1}^I \prod_{k=I-i}^{m-2} \widehat{h}_k^I \ \widehat{f}_{m-1}^I \left[\frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2 \ \frac{\widehat{\alpha}_{j,m,k}}{\sum_{l=0}^{I-k-1} R_{l,k}} \right] \Bigg\} \\
= \ & \sum_{j,m=I-i+1}^{J} (\widehat{X}_{i,j}^{Pa})^I (\widehat{X}_{i,m}^{Pa})^I \left[\frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2 \ \frac{\widehat{\alpha}_{j,m,k}}{\sum_{l=0}^{I-k-1} R_{l,k}} \right]. \qquad (73)
\end{aligned}
$$

This provides Result 4.3.

10 PARAMETER ESTIMATION ERROR, AGGREGATE ACCIDENT YEARS

From Lemma 6.1 and (43) we see that we need to study

$$\Delta_{i,n} \stackrel{def.}{=} \Delta_i^{1/2} \Delta_n^{1/2}, \qquad (74)$$

where Δ_i is given in (66). As above we estimate $\Delta_{i,n}$ using the conditional resampling approach and the corresponding resampling measure $P_{\mathcal{D}_I}^*$, that is, the covariance terms are estimated by

$$R_{i,I-i} \ R_{n,I-n} \ \mathrm{Cov}_{\mathcal{D}_I}^*\left(\Delta_i^{1/2}, \Delta_n^{1/2}\right).$$

This implies that we need to calculate the following four terms: For $i < n$ terms 1 and 2 are given by:

$$\mathrm{Cov}_{\mathcal{D}_I}^*\left(\widehat{f}_{I-i}^I, \widehat{f}_{I-n}^I\right) = \mathrm{Cov}_{\mathcal{D}_I}^*\left(\widehat{f}_{I-n}^I, \sum_{j=I-i+2}^{J} Z_{I-i,j-1}\right) = 0.$$

However, the terms 3 and 4, that are given by

$$\mathrm{Cov}_{\mathcal{D}_I}^*\left(\widehat{f}_{I-i}^I, \sum_{j=I-n+2}^{J} Z_{I-n,j-1}\right) \quad \text{and} \quad \mathrm{Cov}_{\mathcal{D}_I}^*\left(\sum_{j=I-i+2}^{J} Z_{I-i,j-1}, \sum_{m=I-n+2}^{J} Z_{I-n,m-1}\right),$$

require more analysis.

We start with term 3: note that

$$
\begin{aligned}
&\mathrm{Cov}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}, \sum_{j=I-n+2}^{J} Z_{I-n,j-1}\right) \\
&\quad = \mathrm{Cov}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}, Z_{I-n,I-i}\right) + \sum_{j=I-i+2}^{J} \mathrm{Cov}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}, Z_{I-n,j-1}\right) \\
&\quad = \prod_{k=I-n}^{I-i-1} h_k \left[\delta_{I-i}\ \mathrm{Var}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}\right) + \sum_{j=I-i+2}^{J} \delta_{I-i}\ \mathrm{Cov}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}, \widehat{h}^I_{I-i}\right) \prod_{k=I-i+1}^{j-2} h_k\ f_{j-1}\right] \\
&\quad = \prod_{k=I-n}^{I-i-1} h_k \left[\delta_{I-i}\ \frac{\sigma^{1,1}_{I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{j=I-i+2}^{J} \delta_{I-i}\ \frac{\sigma^{1,2}_{I-i} - \sigma^{1,1}_{I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} \prod_{k=I-i+1}^{j-2} h_k\ f_{j-1}\right].
\end{aligned}
$$

Therefore we estimate this term by

$$
R_{i,I-i} R_{n,I-n} \widehat{\mathrm{Cov}}^*_{\mathcal{D}_I}\left(\widehat{f}^I_{I-i}, \sum_{j=I-n+2}^{J} Z_{I-n,j-1}\right) = (\widehat{X}^{Pa}_{i,I-i+1})^I \sum_{j=I-i+1}^{J} (\widehat{X}^{Pa}_{n,j})^I \delta_{I-i} \frac{\widehat{\alpha}_{I-i+1,j,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}}.
$$

So finally, there remains term 4 which satisfies

$$
\begin{aligned}
&\mathrm{Cov}^*_{\mathcal{D}_I}\left(\sum_{j=I-i+2}^{J} Z_{I-i,j-1}, \sum_{m=I-n+2}^{J} Z_{I-n,m-1}\right) \qquad (75) \\
&\quad = \sum_{j,m=I-i+2}^{J} \mathrm{Cov}^*_{\mathcal{D}_I}\left(Z_{I-i,j-1}, Z_{I-n,m-1}\right) + \mathrm{Cov}^*_{\mathcal{D}_I}\left(\sum_{j=I-i+2}^{J} Z_{I-i,j-1}, Z_{I-n,I-i}\right).
\end{aligned}
$$

The second term on the right-hand side of (75) is estimated by

$$
\begin{aligned}
&R_{i,I-i} R_{n,I-n} \widehat{\mathrm{Cov}}^*_{\mathcal{D}_I}\left(\sum_{j=I-i+2}^{J} Z_{I-i,j-1}, Z_{I-n,I-i}\right) \\
&\quad = (\widehat{X}^{Pa}_{n,I-i+1})^I \sum_{j=I-i+2}^{J} (\widehat{X}^{Pa}_{i,j})^I \delta_{I-i}\ \frac{\widehat{\alpha}_{j,I-i+1,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}}.
\end{aligned}
$$

Moreover, for the first term on the right-hand side of (75), we first treat the case $j = m$

$$\begin{aligned}
\mathrm{Cov}^*_{\mathcal{D}_I}\left\{ Z_{I-i,j-1}, Z_{I-n,j-1} \right\} = \prod_{k=I-n}^{I-i-1} h_k \Bigg[\mathrm{Var}^*_{\mathcal{D}_I}\left\{ \prod_{k=I-i}^{j-2} \widehat{h}^I_k \ \widehat{f}^I_{j-1} \right\} \\
-h^2_{I-i} \mathrm{Cov}^*_{\mathcal{D}_I}\Bigg\{ \prod_{k=I-i+1}^{j-2} \widehat{h}^I_k \ \widehat{f}^I_{j-1}, \\
\prod_{k=I-i+1}^{j-2} \Big((1-\delta_k) \ \widehat{h}^I_k + \delta_k \ h_k \Big) \Big((1-\delta_{j-1}) \ \widehat{f}^I_{j-1} + \delta_{j-1} \ f_{j-1} \Big) \Bigg\} \\
-\mathrm{Cov}^*_{\mathcal{D}_I}\Bigg\{ \prod_{k=I-i}^{j-2} \widehat{h}^I_k \ \widehat{f}^I_{j-1}, \\
\prod_{k=I-i}^{j-2} \Big((1-\delta_k) \ \widehat{h}^I_k + \delta_k \ h_k \Big) \Big((1-\delta_{j-1}) \ \widehat{f}^I_{j-1} + \delta_{j-1} \ f_{j-1} \Big) \Bigg\} \\
+h^2_{I-i} \mathrm{Var}^*_{\mathcal{D}_I}\Bigg\{ \prod_{k=I-i+1}^{j-2} \Big((1-\delta_k) \ \widehat{h}^I_k + \delta_k \ h_k \Big) \Big((1-\delta_{j-1}) \ \widehat{f}^I_{j-1} + \delta_{j-1} \ f_{j-1} \Big) \Bigg\} \Bigg].
\end{aligned}$$

Similar as in (71) we obtain the following estimator for $j = m$

$$\prod_{k=I-n}^{I-i-1} \widehat{h}^I_k \left[\prod_{k=I-i}^{j-2} \widehat{h}^I_k \ \widehat{f}^I_{j-1} \right]^2 \left[\delta_{I-i} \ \frac{\widehat{\alpha}_{j,j,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{j-1} \delta^2_k \ \frac{\widehat{\alpha}_{j,j,k}}{\sum_{l=0}^{I-k-1} R_{l,k}} \right],$$

and for $j > m$ or $m > j$ we obtain similarly the estimator

$$\prod_{k=I-n}^{I-i-1} \widehat{h}^I_k \prod_{k=I-i}^{j-2} \widehat{h}^I_k \ \widehat{f}^I_{j-1} \prod_{k=I-i}^{m-2} \widehat{h}^I_k \ \widehat{f}^I_{m-1} \left[\delta_{I-i} \ \frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta^2_k \ \frac{\widehat{\alpha}_{j,m,k}}{\sum_{l=0}^{I-k-1} R_{l,k}} \right].$$

Collecting all the terms we obtain the following covariance terms for the estimation error,

$i < n$,

$$
\begin{aligned}
& R_{i,I-i} R_{n,I-n} \widehat{\Delta}_{i,n} \qquad (76) \\
= \; & (\widehat{X}^{Pa}_{i,I-i+1})^I \sum_{j=I-i+1}^{J} (\widehat{X}^{Pa}_{n,j})^I \delta_{I-i} \frac{\widehat{\alpha}_{I-i+1,j,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} \\
& + (\widehat{X}^{Pa}_{n,I-i+1})^I \sum_{j=I-i+2}^{J} (\widehat{X}^{Pa}_{i,j})^I \delta_{I-i} \frac{\widehat{\alpha}_{j,I-i+1,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} \\
& + \sum_{j,m=I-i+2}^{J} (\widehat{X}^{Pa}_{i,j})^I (\widehat{X}^{Pa}_{n,m})^I \left[\delta_{I-i} \frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2 \frac{\widehat{\alpha}_{j,m,k}}{\sum_{l=0}^{I-k-1} R_{l,k}} \right] \\
= \; & \sum_{j,m=I-i+1}^{J} (\widehat{X}^{Pa}_{i,j})^I (\widehat{X}^{Pa}_{n,m})^I \left[\delta_{I-i} \frac{\widehat{\alpha}_{j,m,I-i}}{\sum_{l=0}^{i-1} R_{l,I-i}} + \sum_{k=I-i+1}^{\min\{j,m\}-1} \delta_k^2 \frac{\widehat{\alpha}_{j,m,k}}{\sum_{l=0}^{I-k-1} R_{l,k}} \right].
\end{aligned}
$$

This and Result 4.3 provides Result 4.6.

11 DATA

i/j	0	1	2	3	4	5	6	7	8	9
0	1'216'632	1'347'072	1'786'877	2'281'606	2'656'224	2'909'307	3'283'388	3'587'549	3'754'403	3'921'258
1	798'924	1'051'912	1'215'785	1'349'939	1'655'312	1'926'210	2'132'833	2'287'311	2'567'056	
2	1'115'636	1'387'387	1'930'867	2'177'002	2'513'171	2'931'930	3'047'368	3'182'511		
3	1'052'161	1'321'206	1'700'132	1'971'303	2'298'349	2'645'113	3'003'425			
4	808'864	1'029'523	1'229'626	1'590'338	1'842'662	2'150'351				
5	1'016'862	1'251'420	1'698'052	2'105'143	2'385'339					
6	948'312	1'108'791	1'315'524	1'487'577						
7	917'530	1'082'426	1'484'405							
8	1'001'238	1'376'124								
9	841'930									

Figure 6: Observed cumulative payments $C^{Pa}_{i,j}$.

i/j	0	1	2	3	4	5	6	7	8	9
0	3'362'115	5'217'243	4'754'900	4'381'677	4'136'883	4'094'140	4'018'736	3'971'591	3'941'391	3'921'258
1	2'640'443	4'643'860	3'869'954	3'248'558	3'102'002	3'019'980	2'976'064	2'946'941	2'919'955	
2	2'879'697	4'785'531	4'045'448	3'467'822	3'377'540	3'341'934	3'283'928	3'257'827		
3	2'933'345	5'299'146	4'451'963	3'700'809	3'553'391	3'469'505	3'413'921			
4	2'768'181	4'658'933	3'936'455	3'512'735	3'385'129	3'298'998				
5	3'228'439	5'271'304	4'484'946	3'798'384	3'702'427					
6	2'927'033	5'067'768	4'066'526	3'704'113						
7	3'083'429	4'790'944	4'408'097							
8	2'761'163	4'132'757								
9	3'045'376									

Figure 7: Observed claims incurred $C_{i,j}^{In}$.

	0	1	2	3	4	5	6	7	8
$\widehat{f}_j^I$	0.1174	0.0922	0.1114	0.1764	0.2424	0.3002	0.3271	0.4279	0.8923
$\widehat{g}_j^I$	0.9761	-0.1896	-0.2026	-0.0802	-0.0501	-0.0663	-0.0564	-0.0548	-0.1077
$\widehat{h}_j^I$	1.8586	0.7182	0.6860	0.7434	0.7075	0.6335	0.6165	0.5173	0.0000
$\widehat{\sigma}_j^{1,1}$	4'241	5'560	5'103	2'796	16'724	9'625	18'536	26	0.04
$\widehat{\sigma}_j^{2,2}$	48'855	10'044	11'535	856	300	1'025	567	345	210
$\widehat{\sigma}_j^{1,2}$	1'931	2'771	1'403	-175	-47	-895	-3'130	-95	

Figure 8: Estimated parameters, see Table 3 in Dahms [3].

References

[1] AISAM-ACME (2007). AISAM-ACME study on non-life long tail liabilities. Reserve risk and risk margin assessment under Solvency II. October 17, 2007.

[2] Bühlmann, B., De Felice, M., Gisler A., Moriconi, F., Wüthrich, M.V. (2009). Recursive credibility formula for chain ladder factors and the claims development result. ASTIN Bulletin 39/1, 275-306.

[3] Dahms, R. (2008). A loss reserving method for incomplete data. Bulletin Swiss Assocation Actuaries, 2008/1&2, 127-148.

[4] Halliwell, L.J. (1997). Conjoint prediction of paid and incurred losses. CAS Forum (Summer), Vol. 1, 241-379.

[5] Mack, T. (1993). Distribution-free calculation of the standard error of chain ladder reserve estimates. ASTIN Bulletin 23/2, 213-225.

[6] Mack, T. (1997). Schadenversicherungsmathematik. Schriftenreihe Angewandte Versichungsmathematik 28, 230-234.

[7] Merz, M., Wüthrich, M.V. (2008). Modelling the claims development result for solvency purposes. CAS E-Forum, Fall 2008, 542-568.

[8] Ohlsson, E., Lauzeningks, J. (2008). The one-year non-life insurance risk. Conference Paper, ASTIN Colloquium 2008, Manchester.

[9] Quarg, G., Mack, T. (2004). Munich chain ladder. Blätter DGVFM XXVI, 597-630.

[10] Wüthrich, M.V., Merz, M. (2008). Stochastic Claims Rerserving Methods in Insurance. Wiley.