Chain-Ladder method and midyear loss reserving

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Abstract

Although, loss reserving has been deeply studied in the literature there are still practical issues that have not been addressed a lot. One of them is the estimation of reserves during the year, which is necessary for forecasts or closings during the year. We will study the following question: What can be done for forecasts and closings during the year that goes along with the reserving at year end? In order to make it not too complicated we will focus on the Chain-Ladder method introduced by Mack [4]. We will describe several methods that are used in practice. We will discuss advantages and disadvantages of these methods based on a simple deterministic example. Roughly spoken we will see that you may shift development or accident periods; or may split development periods, but should not split accident periods.

KEYWORDS: STOCHASTIC RESERVING, CHAIN LADDER, MEAN SQUARED ERROR OF PREDICTION, SOLVENCY RESERVING RISK, CLAIMS DEVELOPMENT RESULT, MIDYEAR RESERVING.

1 Introduction

Assume we have a portfolio for which we believe that the Chain-Ladder method, see [4], is applicable in order to estimate the corresponding loss reserves at the end of a year. That means we have cumulative payments (or incurred losses) $C_{i,k}$ for each accident year $0 \leq i \leq I$ and each development year $0 \leq k \leq I$ and we assume that there exist constants f_k and σ_k^2 such that

- $\mathrm{E}\left[C_{i,k+1} \middle| C_{i,0}, \ldots, C_{i,k}\right] = f_k C_{i,k}$
- $\mathrm{Var}\left[C_{i,k+1} \middle| C_{i,0}, \ldots, C_{i,k}\right] = \sigma_k^2 C_{i,k}$
- accident years are independent, i.e. the vectors $(C_{i,0}, \cdots, C_{i,I})_{0 \leq i \leq I}$ are independent.

At the end of year I all payments $C_{i,k}$ with $i + k \leq I$ are known and we have to estimate the future payments $C_{i,k}$ with $i + k > I$, see Figure 1.

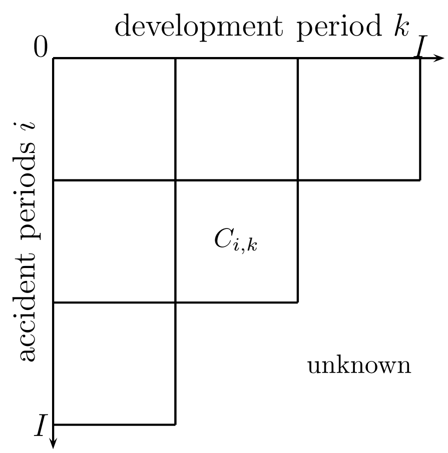

Figure 1: Claims development triangle.

Under this assumptions the classical Chain-Ladder estimates for future payments $C_{i,k}$, $i + k > I$, are

$$\hat{C}_{i,k} := \widehat{f}_{k-1} \cdot \cdots \cdot \widehat{f}_{I-i} C_{i,I-i}.$$

Although, in practice the development factors f_k are often estimated by a manual weighted (actuarial judgement) average of the observed development factors, we will use the variance minimizing weights in this paper, i.e.

$$\widehat{f}_k := \sum_{i=0}^{I-k-1} \frac{C_{i,k}}{\sum_{h=0}^{I-k-1} C_{h,k}} \frac{C_{i,k+1}}{C_{i,k}} = \frac{\sum_{i=0}^{I-k-1} C_{i,k+1}}{\sum_{i=0}^{I-k-1} C_{i,k}}.$$

Therefore, we know how to estimate the reserves at the end of the year. But what should we do during the year. There are two typical situations:

- half-year, quarterly or even monthly closings and
- forecasts of the annual closing, for instance at the end of November.

In both cases we would like to compare our actuarial judgement with the one of the previous annual closing and its consequences in terms of the claims development result.

In order to make discussions easier let us look at the simple example of cumulative annual payments presented in Figure 2, which is even deterministic in its development.

100	250	350
200	500	700
260	650	910

f_0=2.5 f_1=1.4

Figure 2: Situation at the end of December.

Now assume further six months have passed. Then we have the situation shown in Figure 3.

100	250	350	350
200	500	650	
260	455		
75			

Figure 3: Situation at the end of June.

Here the grey cells contain the new information obtained during the last six months. Obviously we cannot apply the Chain-Ladder method on this triangle, because the last diagonal contains incomplete data. Therefore, let us assume we have more granular data $\widetilde{C}_{i,k}$ for each accident half-year i and each development half-year k, see Figure 4[1]. Both data are related via $C_{i,k} = \widetilde{C}_{2i,2k+1} + \widetilde{C}_{2i+1,2k}$.

	12	24	36	48
0	25 75 100 25	100 150 250 75 100	175 175 350 150 175	175 350 175
1	50 150 200 50	200 300 500 150 200	350 650 300	
2	65 195 260 65	260 455 195		
3	75 75			

Figure 4: Half-year data.

For better illustration in figures and in contrast to mathematical formulas we will label accident periods with years (or half-years) whereas development periods are labelled with the number of months elapsed since the begin of the corresponding accident period.

In the following sections we will discuss several methods for the midyear reserving, which are used in practice. We will base our discussion on the presented very simple example and will look at the following topics:

Theoretical consistency: Is the method theoretically in line with the Chain-Ladder assumptions at year end?

Discussion of CDR: To what extent can we discus the claims development result?

Rolling forward: To what extent can we use our actuarial judgement of the previous annual closing for the analysis at the end of June?

[1] In figure 4 numbers in large letters correspond to the yearly triangle and numbers in small letters to half-year data. So in each square the sum of the small figures on the left corresponds to end of June and the sum of the small figures on the right to end of December, which equals the figure in the middle of the square, except for the last diagonal, where we only see data up to end of June.

Closing: For a closing at some date we need an estimate of future payments for all claims happened up to that data and payments for all other claims must be excluded from the estimate. That leads to the question: Can the methods be used for a closing at the end of June?

Forecast: For a forecast of the next annual closing at some date we need an estimate of future payments for all claims that already have happened up that date or will happen up to the end of the year. That leads to the question: Can the methods be used at the end of June in order to estimate a forecast of the next annual closing?

Generalisation: How easily can the method be generalised to other months than June?

Usability: Are there situations in practice where the method might be beneficial?

Estimation of uncertainties : Can the formulas of Mack, see [4], Merz-Wüthrich, see [5], Röhr, see [6] or other authors for the ultimate and the solvency (one-year) uncertainty be generalised for the midyear estimation of reserves?

In the following section we will introduce ways to look at the data, such that at the end of June the Chain-Ladder method can be applied to project future payments either for a forecast or a closing. Regarding the estimation of uncertainties we have here to distinguish two situations:

- **Forecast:** Since we project claims and events that have not happened yet and therefore belong to the premium risk, we believe that the approaches of Mack, Merz-Wüthrich or Röhr should not be used.
- **Closing:** In some cases its is possible to adapt the approaches of Mack, Merz-Wüthrich or Röhr or to use the more general approach of [3] in order to couple triangles. Nevertheless, we believe it is better to analyse the decay of uncertainties using the approach of Röhr based on annual data and interpolate for the midyear reserving. The reasons are:
  - We believe that such uncertainties, in terms of coefficient of variation, should not vary a lot over time,
  - in practice the corresponding estimators often do vary a lot over time and
  - actuarial judgement has to be used.

Therefore, we will not investigate the estimation of uncertainties further in this paper.

Before we will discuss the methods let us make some remarks about our preference of the annual data regarding the Chain-Ladder assumptions, or in other words: Why do we not start with more granular data, like half-year or even monthly data? We all know that the Chain-Ladder assumptions, in particular the assumption of independent accident periods, is very strong and almost never strictly fulfilled in practice, even for annual triangles. If we look at more granular data there are seasonal influences, which may distort the development even more. Besides the weather there are a lot of other seasonal effects, for instance in some years Easter is in March and in others years in April, in some years Christmas and New Year are on a weekend in other years not, school holidays may slightly change from year to year etc. Those seasonal effects are more likely to go along with the Chain-Ladder assumptions for annual data than for more granular data. Therefore, we assume that the annual data $C_{i,k}$ fulfil the Chain-Ladder assumptions. In section 2.1 we will present a heuristic test that may be used to decide if data with more granular development periods can be used, too, and in section 2.4 we will show that it is very unlikely that data with more granular accident periods still fulfil the Chain-Ladder assumptions. That means: You may split development periods but you should not split accident periods!

2 Methods for midyear reserving

In the following sections we will look at several methods that might be used at the end of June. Therefore, we will always assume that accident periods are independent. The corresponding data will always be denoted by $\bar{C}_{i,k}$. Moreover, $C_{i,k}$ and $\widetilde{C}_{i,k}$ will always refer to the annual and half-year data, respectively, see figure 4. So $\bar{C}_{i,k}$ vary from method to method, whereas $C_{i,k}$ and $\widetilde{C}_{i,k}$ are fixed.

2.1 Splitting of development periods

2.1.1 Method description

This method splits development periods into two or more subperiods, see Figure 5.

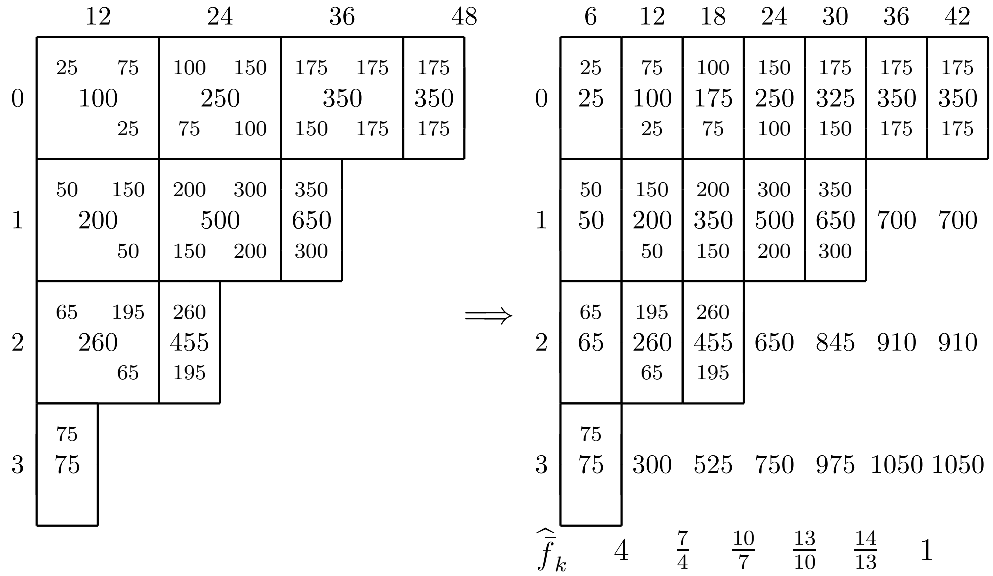

Figure 5: Splitting of development periods.

Here we have

$$\bar{C}_{i,k} := \widetilde{C}_{2i,k} + \widetilde{C}_{2i+1,k-1}, \qquad \text{where we define } \widetilde{C}_{i,-1} := 0.$$

Then the annual data fulfil

$$C_{i,k} = \bar{C}_{i,2k+1}.$$

2.1.2 Theoretical consistency

It is a well know fact that if $\bar{C}_{i,k}$ satisfies the Chain-Ladder assumptions then $C_{i,k}$ will do so, too. Nevertheless, in order to get used to the framework let us briefly restate this fact in an even more general form.

Lemma 2.1 *Assume the $\bar{C}_{i,k}$ fulfil the Chain-Ladder assumptions with development factors $\bar{f}_k$ and variance parameters $\bar{\sigma}_k^2$, $0 \le k < I$, and let $0 \le k_0 < \cdots < k_n < I$ be a subsequence of the development indices. Then*

$$C_{i,j} := \bar{C}_{i,k_j}$$

fulfil the Chain-Ladder assumptions with development factors

$$f_j := \bar{f}_{k_j} \cdot \ldots \cdot \bar{f}_{k_{j+1}-1}$$

and variance parameters

$$\sigma_j^2 := \sum_{k=k_j}^{k_{j+1}-1} \bar{f}_{k_j} \cdot \ldots \cdot \bar{f}_{k-1} \bar{\sigma}_k^2 \bar{f}_{k+1}^2 \cdot \ldots \cdot \bar{f}_{k_{j+1}-1}^2, \qquad 0 \le j \le n.$$

Proof: Clearly, the property of independent accident periods has not changed. For the properties on the first moment we get

$$\begin{aligned} \mathrm{E}\left[C_{i,j+1} \middle| C_{i,0}, \ldots, C_{i,j}\right] &= \mathrm{E}\left[\mathrm{E}\left[\bar{C}_{i,k_{j+1}} \middle| \bar{C}_{i,0}, \ldots, \bar{C}_{i,k_j}\right] \middle| C_{i,0}, \ldots, C_{i,j}\right] \\ &= \bar{f}_{k_j} \cdot \ldots \cdot \bar{f}_{k_{j+1}-1} \mathrm{E}\left[\bar{C}_{i,k_j} \middle| C_{i,0}, \ldots, C_{i,j}\right] = \bar{f}_{k_j} \cdot \ldots \cdot \bar{f}_{k_{j+1}-1} C_{i,j} =: f_j C_{i,j}. \end{aligned} \tag{1}$$

Analogously, one can show that

$$\mathrm{Var}\left[C_{i,j+1} \middle| C_{i,0}, \ldots, C_{i,j}\right] = \sum_{k=k_j}^{k_{j+1}-1} \bar{f}_{k_j} \cdot \ldots \cdot \bar{f}_{k-1} \bar{\sigma}_k^2 \bar{f}_{k+1}^2 \cdot \ldots \cdot \bar{f}_{k_{j+1}-1}^2 C_{i,j} =: \sigma_j^2 C_{i,j},$$

where we set the product over an empty set of factors to one. □

If for all development periods $k_j, \ldots, k_{j+1}-1$ the number of observed accident periods are the same, then the estimated development factors satisfy

$$\widehat{\bar{f}}_{k_j} \cdot \ldots \cdot \widehat{\bar{f}}_{k_{j+1}-1} = \widehat{f}_j, \tag{2}$$

but for the standard estimators for the variance parameters we get

$$\begin{aligned} &\mathrm{E}\left[\sum_{k=k_j}^{k_{j+1}-1} \widehat{\bar{f}}_{k_j} \cdot \ldots \cdot \widehat{\bar{f}}_{k-1} \widehat{\bar{\sigma}}_k^2 \widehat{\bar{f}}_{k+1}^2 \cdot \ldots \cdot \widehat{\bar{f}}_{k_{j+1}-1}^2\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\sum_{k=k_j}^{k_{j+1}-1} \widehat{\bar{f}}_{k_j} \cdot \ldots \cdot \widehat{\bar{f}}_{k-1} \widehat{\bar{\sigma}}_k^2 \widehat{\bar{f}}_{k+1}^2 \cdot \ldots \cdot \widehat{\bar{f}}_{k_{j+1}-1}^2 \middle| \mathcal{D}_{k_{j+1}-1}\right]\right] \\ &= \mathrm{E}\left[\sum_{k=k_j}^{k_{j+1}-1} \widehat{\bar{f}}_{k_j} \cdot \ldots \cdot \widehat{\bar{f}}_{k-1} \widehat{\bar{\sigma}}_k^2 \widehat{\bar{f}}_{k+1}^2 \cdot \ldots \cdot \widehat{\bar{f}}_{k_{j+1}-2}^2 \mathrm{E}\left[\widehat{\bar{f}}_{k_{j+1}-1}^2 \middle| \mathcal{D}_{k_{j+1}-1}\right]\right] \\ &= \mathrm{E}\left[\sum_{k=k_j}^{k_{j+1}-1} \widehat{\bar{f}}_{k_j} \cdot \ldots \cdot \widehat{\bar{f}}_{k-1} \widehat{\bar{\sigma}}_k^2 \widehat{\bar{f}}_{k+1}^2 \cdot \ldots \cdot \widehat{\bar{f}}_{k_{j+1}-2}^2\right] \left(\frac{\bar{\sigma}_{k_{j+1}-1}^2}{\sum_{i=0}^{I-k_{j+1}} \bar{C}_{i,k_{j+1}-1}} + \bar{f}_{k_{j+1}-1}^2\right) \\ &\ge \mathrm{E}\left[\sum_{k=k_j}^{k_{j+1}-1} \widehat{\bar{f}}_{k_j} \cdot \ldots \cdot \widehat{\bar{f}}_{k-1} \widehat{\bar{\sigma}}_k^2 \widehat{\bar{f}}_{k+1}^2 \cdot \ldots \cdot \widehat{\bar{f}}_{k_{j+1}-2}^2\right] \bar{f}_{k_{j+1}-1}^2 \ge \ldots \ge \mathrm{E}\left[\widehat{\sigma}_j^2\right] = \sigma_j^2, \end{aligned} \tag{3}$$

where $\mathcal{D}_k := \sigma(C_{i,j} \colon 0 \le i \le I, 0 \le j \le k)$ represents all the information up to development period k. Equality in (3) holds if and only if we are in the deterministic case where all variance parameters are equal to zero. Therefore, (3) may be used as test for the Chain-Ladder assumptions of $\bar{C}_{i,k}$.

In practice one often observes

$$\sum_{k=k_j}^{k_{j+1}-1} \widehat{\bar{f}}_{k_j} \cdot \ldots \cdot \widehat{\bar{f}}_{k-1} \widehat{\bar{\sigma}}_k^2 \widehat{\bar{f}}^2_{k+1} \cdot \ldots \cdot \widehat{\bar{f}}^2_{k_{j+1}-1} \ll \widehat{\sigma}_j^2.$$

That means, in such situations it is very unlikely that the more granular triangle $\bar{C}_{i,k}$ fulfil the Chain-Ladder assumptions. Nevertheless, since the estimated reserves are still the same as those based on the annual triangle, we still might use the more granular triangle for reserving, but we should not use it to estimate the corresponding mean squared error of prediction.

2.1.3 Rolling forward

Beside the CDR we also can compare annual link ratios and corresponding estimated development factors directly with the corresponding products at midyear, see (1). This is very helpful for the explanation of the CDR.

2.1.4 Closing

At the end of June the estimates for the last accident period contain a forecast of the ultimate of the second half-year. Therefore, the method cannot be applied directly for midyear closings. Nevertheless, for most portfolios there is a very stable "earning pattern" that can be used to split the estimated ultimate for the last accident year into the part that belongs to claims already happened and the part that belongs to the forecast. But you never should do that automatically, because even very stable earning pattern may lead to strange accruals, for instance in the case of large claims already known.

2.1.5 Forecast

The method automatically leads to a forecast of the annual figures at the end of June.

2.1.6 Generalisation

If we want to apply this method to other dates we could split the development periods accordantly, for instance for an evaluation at the end of November we could split into January-November and December data. But doing so we loose the comparability of the (estimated) development factors, see (1) and (2), with respect to our evaluation at the end of June. If we want to keep this comparability we have to look at even more granular data, which may finally lead to an analyse of monthly development periods. Therefore, we can generalise this method to be used at the end of other months. But this goes along with less usability: Imagine a portfolio that takes 30 or more years until everything is paid then for a forecast at the end of November we have to estimate at least 359 development factors!

2.1.7 Usability

Doubling the number of development periods for an analysis at the end of June may still be OK, but handling quarterly or even monthly forecasts will be very hard. If we have to estimate more than one hundred development factors we might overlook important changes in those pattern.

2.2 Shifting of development periods

2.2.1 Method description

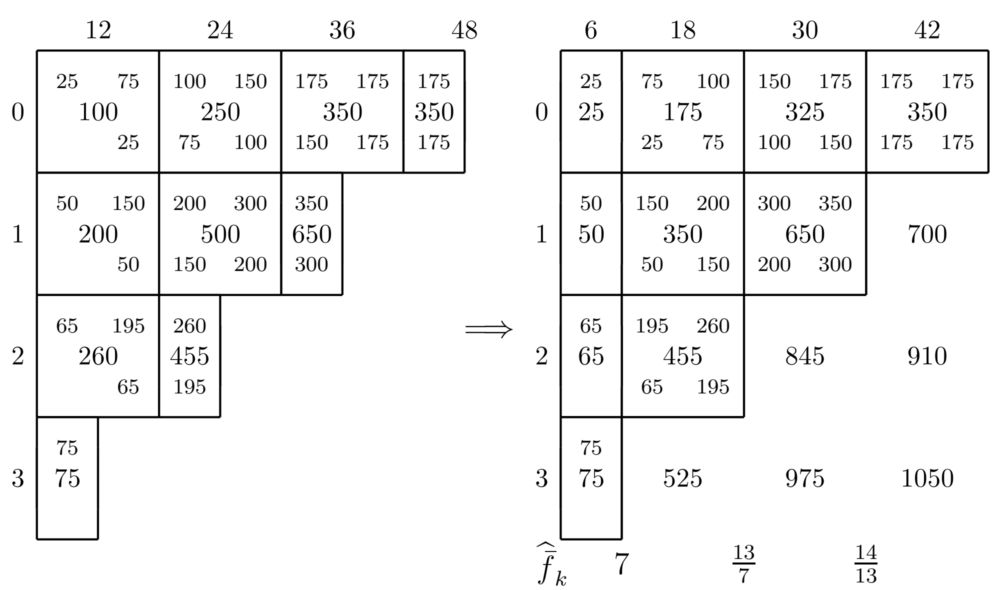

Figure 6: Shifting of development periods.

In section 2.1 we have seen that splitting of development periods is problematic, because the number of parameters increases rapidly and we even may get some theoretical problems in cases where the variance test (3) fails. Surprisingly, there is a very simple solution to both problems, but it comes with some costs regarding the rolling forward.

The idea is to stay with development periods of twelve months, except for the first development period, where we only look at six months at the end of June, see Figure 6. Here we have

$$\bar{C}_{i,k} := \widetilde{C}_{2i,2k} + \widetilde{C}_{2i+1,2k-1}, \qquad \text{where we define } \widetilde{C}_{i,-1} := 0.$$

2.2.2 Theoretical consistency

We have seen in section 2.1, see Lemma 2.1, that if the more granular data with six months grouped into one development period fulfil the Chain-Ladder assumptions, then the annual data $C_{i,k}$ and the shifted data $\bar{C}_{i,k}$ will do the same. Moreover, since there is no direct connection between $C_{i,k}$ and $\bar{C}_{i,k}$, there might be other stochastic models such that both $C_{i,k}$ and $\bar{C}_{i,k}$ fulfil the Chain-Ladder assumptions. A heuristic indication for such models is the behaviour of the estimated variance parameters and of the mean squared errors of prediction (MSEP). In practice we often observe that even if the variance test (3) fails, which means that the more granular data do not fulfil the Chain-Ladder assumptions, the MSEP estimated based on the shifted data $\bar{C}_{i,k}$ seems to be consistent over time, see Figure 7[2].

2.2.3 Rolling forward

Compared to the method of splitting development periods we still can discuss the CDR, but, because of the shifting, we loose the comparability of link ratios and corresponding estimated development factors.

[2]The curve in figure 7 corresponds to the typical development of the ultimate uncertainty estimated by the method of shifted development periods. This is the behaviour we would expect. The two additional dots representing the often observed ultimate uncertainty estimated at year end based on split development periods, see section 2.1.

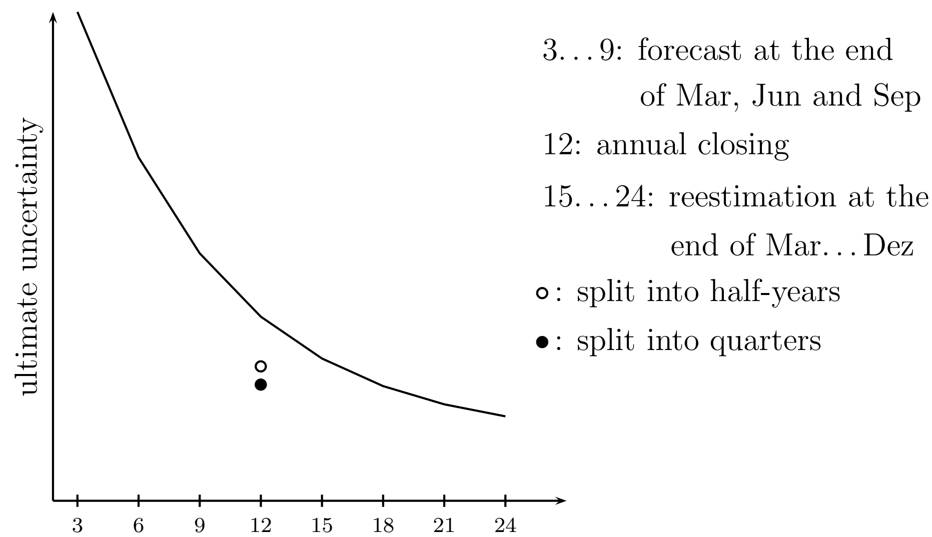

Figure 7: Development of the ultimate uncertainty

2.2.4 Closing

Since the projection for the last accident year contains some forecast, the same arguments as for the splitting of development periods, see section 2.1, apply for shifted development periods. Therefore, this method cannot be used directly to estimate reserves for midyear closings.

2.2.5 Forecast

The method automatically leads to a forecast of the annual figures at the end of June.

2.2.6 Generalisation

This method can easily be generalised to other months. We only have to shift the data by the corresponding number of development months. The corresponding typical development of the ultimate uncertainty is shown by the curve in figure 7.

2.2.7 Usability

The only disadvantage of this method in practice is that we cannot compare link ratios and corresponding estimates of the development factors. This might make the discussion of changes harder. For instance, it sometimes happened that you exclude strange link ratios for the midyear analysis, but at the annual closing those strange link ratios have vanished.

2.3 Extrapolation of the last diagonal

2.3.1 Method description

The main disadvantage of shifting development periods is that we cannot compare link ratios and corresponding estimated development factors with the ones from the last annual closing, which makes the discussion of the claims development result much harder, seen section 2.2. Now we want to use shifted development periods in order to extrapolate the last partly observed diagonal and then apply the Chain-Ladder method on the extrapolated annual triangle, see Figure 8. Doing so, we can compare most of the link ratios and the estimated development factors with the corresponding year-end figures.

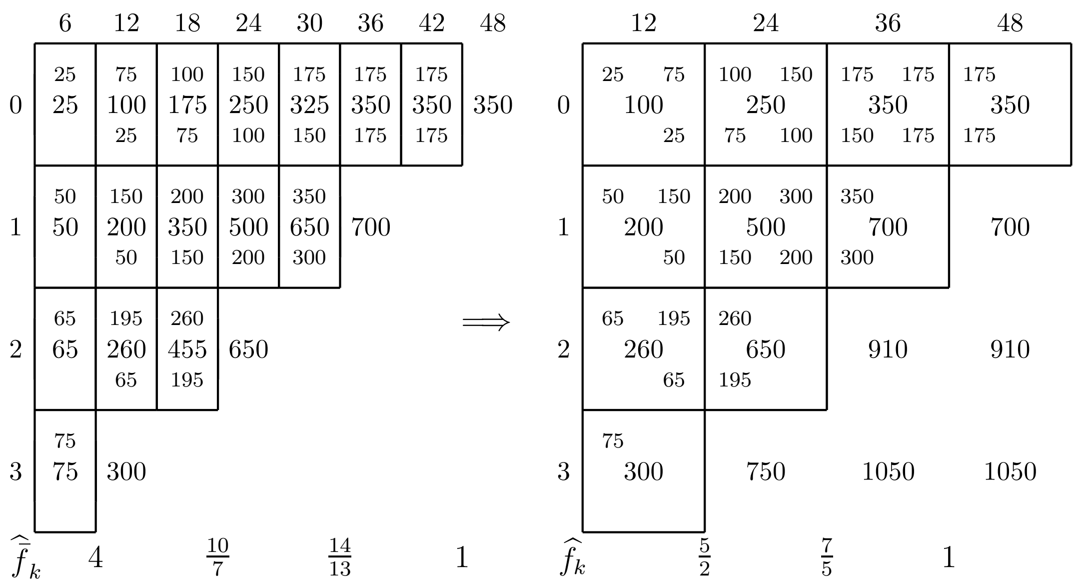

Figure 8: Extrapolation of the last diagonal.

2.3.2 Theoretical consistency

Like in section 2.1 we assume that the half-year data $\bar{C}_{i,k} := \widetilde{C}_{2i,k} + \widetilde{C}_{2i+1,k-1}$ fulfil the Chain-Ladder assumptions. We have seen that in this case the annual data $C_{i,k} := \bar{C}_{i,2k+1}$ also satisfy the Chain-Ladder assumptions.

Following the motivation of first extrapolating the latest half unknown diagonal and than apply the standard Chain-Ladder approach on the resulting annual triangle we estimate the ultimate at the end of June by

$$\hat{C}_{i,I} := \widehat{f}_I \cdots \widehat{f}_{I+1-i}\widehat{\bar{f}}_{2(I+1-i)}\bar{C}_{i,2(I+1-i)}.$$

Note, since we are at the end of June, the data for the latest accident year $\bar{C}_{I+1,0}$ only contain claims happened until the end of June.

Now the question is: How to estimate the parameters? Clearly, in order to get estimates for $\bar{f}_k$ we will use the standard estimators based on the half-year data

$$\widehat{\bar{f}}_k := \frac{\sum_{i=0}^{I-\lfloor k/2 \rfloor} \bar{C}_{i,k+1}}{\sum_{i=0}^{I-\lceil k/2 \rceil} \bar{C}_{i,k}},$$

where $\lfloor k/2 \rfloor$ means to round down $k/2$. Note, since we only have new data for the first half of the most recent year, the estimated development factors for the development from June to December $\widehat{\bar{f}}_{2k+2}$ are the same like at the end of last year, i.e.

$$\widehat{\bar{f}}_{2k+2} = \widehat{\bar{f}}^{ye}_{2k+2} = \frac{\sum_{i=0}^{I-1-k} \bar{C}_{i,2k+3}}{\sum_{i=0}^{I-1-k} \bar{C}_{i,2k+2}}.$$

If we calculate now the standard estimators based on the triangle with the extrapolated last diagonal we get

$$\begin{aligned}
\widehat{f}_k &:= \frac{\sum_{i=0}^{I-k} \hat{C}_{i,k+1}}{\sum_{i=0}^{I-k} C_{i,k}} = \frac{\sum_{i=0}^{I-k-1} \bar{C}_{i,2k+3} + \widehat{\bar{f}}_{2k+2}\bar{C}_{I-k,2k+2}}{\sum_{i=0}^{I-k} C_{i,k}} \\
&= \frac{\widehat{\bar{f}}_{2k+2} \sum_{i=0}^{I-k-1} \bar{C}_{i,2k+2} + \widehat{\bar{f}}_{2k+2}\bar{C}_{I-k,2k+2}}{\sum_{i=0}^{I-k} C_{i,2k+1}} = \widehat{\bar{f}}_{2k+1}\widehat{\bar{f}}_{2k+2}, \qquad (4)
\end{aligned}$$

which means we get the same estimates for the development factors and therefore the same estimates for the ultimate like in the case of split development periods, analysed in section 2.1.

Another way to look at the estimated development factors is as follows:

$$\begin{aligned}
\widehat{f}_k &= \frac{\sum_{i=0}^{I-k} \hat{C}_{i,k+1}}{\sum_{i=0}^{I-k} C_{i,k}} = \frac{\sum_{i=0}^{I-k-1} C_{i,k+1} + \widehat{\bar{f}}_{2k+2}\bar{C}_{I-k,2k+2}}{\sum_{i=0}^{I-k} C_{i,k}} \\
&= \frac{\sum_{i=0}^{I-k-1} C_{i,k}}{\sum_{i=0}^{I-k} C_{i,k}} \widehat{f}^{ye}_k + \left(1 - \frac{\sum_{i=0}^{I-k-1} C_{i,k}}{\sum_{i=0}^{I-k} C_{i,k}}\right) \frac{\bar{C}_{I-k,2k+2}}{\bar{C}_{I-k,2k+1}} \widehat{\bar{f}}_{2k+2}. \qquad (5)
\end{aligned}$$

That means, the half-year estimate of the development factors f_k is a weighted mean of the last year-end estimate

$$\widehat{f}^{ye}_k := \frac{\sum_{i=0}^{I-1-k} C_{i,k+1}}{\sum_{i=0}^{I-1-k} C_{i,k}}$$

and the newly observed first half-year development $\frac{\bar{C}_{I-k,2k+2}}{\bar{C}_{I-k,2k+1}}$ times the estimated development of the second half-year $\widehat{\bar{f}}_{2k+2}$, where the weights are the standard Chain-Ladder weights. Moreover, one can show that these weights are almost the optimal weights in order to approximate in this way the estimates of the development factors

$$\widehat{f}_k^{ye+1} := \frac{\sum_{i=0}^{I-k} C_{i,k+1}}{\sum_{i=0}^{I-k} C_{i,k}}$$

of the next annual closing:

Lemma 2.2 *The weights* α_k *which minimise*

$$\mathrm{E}\left[\left((1-\alpha_k)\widehat{f}_k^{ye} + \alpha_k \frac{\bar{C}_{I-k,2k+2}}{\bar{C}_{I-k,2k+1}} \widehat{\bar{f}}_{2k+2} - \widehat{f}_k^{ye+1}\right)^2 \middle| \bar{C}_{i,j} : i + \lfloor j/2 \rfloor \leq I\right] \tag{6}$$

satisfy

$$\alpha_k := \frac{\bar{f}_{2k+2} C_{I-k,k} + \mathcal{O}\left(\bar{f}_{2k+1} - \widehat{\bar{f}}^{ye}_{2k+1}\right)}{\widehat{\bar{f}}_{2k+2} \sum_{i=0}^{I-k} C_{i,k} + \mathcal{O}\left(\left(\bar{f}_{2k+1} - \widehat{\bar{f}}^{ye}_{2k+1}\right)^2\right)} \approx \frac{C_{I-k,k}}{\sum_{i=0}^{I-k} C_{i,k}} = 1 - \frac{\sum_{i=0}^{I-k-1} C_{i,k}}{\sum_{i=0}^{I-k} C_{i,k}}.$$

Proof: In order to shorten notations we will skip the condition $\left\{\bar{C}_{i,j} : i + \lfloor j/2 \rfloor \leq I\right\}$ in the following derivation. Since terms of variances and expectations are much easier to handle we will use $\mathrm{E}\left[X^2\right] = \mathrm{Var}\left[X\right] + \mathrm{E}\left[X\right]^2$ to calculate (6). Let us start with the expectation. The only two parts that are not measurable with respect to the condition are $\bar{C}_{I-k,2k+2}$ and $C_{I-k,k+1} = \bar{C}_{I-k,2k+3}$, which is part of $\widehat{f}_k^{ye+1}$. Therefore, we get

$$\begin{aligned}
&\mathrm{E}\left[(1-\alpha_k)\widehat{f}_k^{ye} + \alpha_k \frac{\bar{C}_{I-k,2k+2}}{\bar{C}_{I-k,2k+1}} \widehat{\bar{f}}_{2k+2} - \widehat{f}_k^{ye+1}\right] \\
&= (1-\alpha_k)\widehat{\bar{f}}^{ye}_{2k+1}\widehat{\bar{f}}^{ye}_{2k+2} + \alpha_k \bar{f}_{2k+1}\widehat{\bar{f}}_{2k+2} - \frac{\sum_{i=0}^{I-k-1} C_{i,k+1} + \bar{f}_{2k+1}\bar{f}_{2k+2} C_{I-k,k}}{\sum_{i=0}^{I-k} C_{i,k}}.
\end{aligned}$$

Since $\widehat{\bar{f}}^{ye}_{2k+2} = \widehat{\bar{f}}_{2k+2}$ we can proceed with

$$= \alpha_k \widehat{\bar{f}}_{2k+2}\left(\bar{f}_{2k+1} - \widehat{\bar{f}}^{ye}_{2k+1}\right) + \text{const.}$$

Analogously, we calculate the variance

$$\begin{aligned}
&\operatorname{Var}\left[(1-\alpha_k)\widehat{f}_k^{ye} + \alpha_k \frac{\bar{C}_{I-k,2k+2}}{\bar{C}_{I-k,2k+1}}\widehat{\bar{f}}_{2k+2} - \widehat{f}_k^{ye+1}\right] \\
&= \frac{\alpha_k^2 \widehat{\bar{f}}_{2k+2}^2 \operatorname{Var}\left[\bar{C}_{I-k,2k+2}\right]}{\bar{C}_{I-k,2k+1}^2} - 2\frac{\alpha_k \widehat{\bar{f}}_{2k+2} \operatorname{Cov}\left[\bar{C}_{I-k,2k+2}, \bar{C}_{I-k,2k+3}\right]}{\bar{C}_{I-k,2k+1}\sum_{i=0}^{I-k} C_{i,k}} + \text{const} \\
&= \alpha_k^2 \frac{\bar{\sigma}_{2k+1}^2}{C_{I-k,k}} \widehat{\bar{f}}_{2k+2}^2 - 2\alpha_k \frac{\widehat{\bar{f}}_{2k+2}\bar{f}_{2k+2}\bar{\sigma}_{2k+1}^2}{\sum_{i=0}^{I-k} C_{i,k}} + \text{const.}
\end{aligned}$$

Differentiating the sum of the variance and the square of the expectation with respect to α_k we get

$$\begin{aligned}
\frac{2\widehat{\bar{f}}_{2k+2}\bar{\sigma}_{2k+1}^2}{C_{I-k,k}\sum_{i=0}^{I-k} C_{i,k}} \Bigg[\alpha_k \Bigg(\widehat{\bar{f}}_{2k+2} \sum_{i=0}^{I-k} C_{i,k} + \mathcal{O}\Bigg(\Big(\bar{f}_{2k+1} - \widehat{\bar{f}}_{2k+1}^{ye}\Big)^2\Bigg)\Bigg) \\
- \bar{f}_{2k+2} C_{I-k,k} + \mathcal{O}\Big(\bar{f}_{2k+1} - \widehat{\bar{f}}_{2k+1}^{ye}\Big)\Bigg].
\end{aligned}$$

Here the $\mathcal{O}$-terms stem from the square of the expectation and from the variance. Setting it equal to zero and solving the resulting equation with respect to α_k we get the stated formula. □

2.3.3 Rolling forward

Besides the CDR we also can compare link ratios and corresponding estimated development factors directly, which is very helpful for the explanation of the CDR.

2.3.4 Closing

At the end of June the estimates for the last accident period contain a forecast of the ultimate of the second half-year. Therefore, the method cannot be applied directly for midyear closings.

2.3.5 Forecast

The method automatically leads to a forecast of the annual figures at the end of June.

2.3.6 Usability

The advantages of extrapolating the latest diagonal compared to the method of spitting development periods, see section 2.1, are

- Even for quarterly and monthly forecasts the number of estimated parameters only double in comparison with the annual closing.
- In most practical cases the estimated variance parameters are more in line with the corresponding estimates of the last annual closing.

2.4 Splitting of accident periods

2.4.1 Method description

The idea of this method is to split accident years. But in contrast to the method of splitting development periods a split of accident periods requires a corresponding split of development periods, see Figure 9.

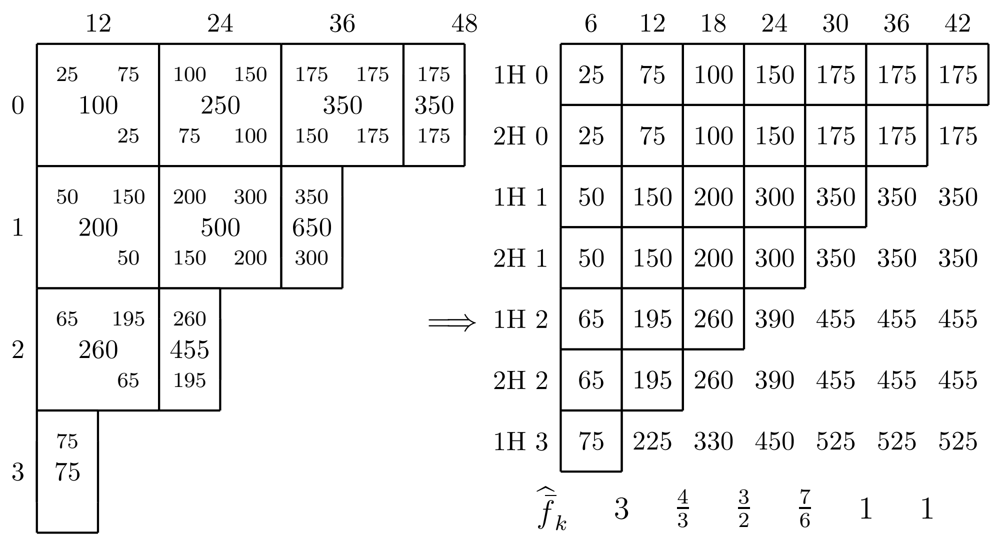

Figure 9: Splitting of accident periods.

Here we have

$$\bar{C}_{i,k} := \widetilde{C}_{i,k}.$$

2.4.2 Theoretical consistency

Assume both, the annual data $C_{i,k}$ and the half-year data $\bar{C}_{i,k}$, simultaneously fulfil the corresponding Chain-Ladder assumptions and define

$$\mathcal{B}_{i,k} := \sigma\left(C_{i,0}, \ldots, C_{i,k}\right) \qquad \text{and} \qquad \bar{\mathcal{B}}_{i,k} := \sigma\left(\bar{C}_{i,0}, \ldots, \bar{C}_{i,k}\right).$$

Then we get

$$\begin{aligned} f_k(\bar{C}_{2i,2k+1} + \bar{C}_{2i+1,2k}) = f_k C_{i,k} &= \mathrm{E}\left[C_{i,k+1} \middle| \mathcal{B}_{i,k}\right] = \mathrm{E}\left[\bar{C}_{2i,2k+3} + \bar{C}_{2i+1,2k+2} \middle| \mathcal{B}_{i,k}\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\bar{C}_{2i,2k+3} + \bar{C}_{2i+1,2k+2} \middle| \bar{\mathcal{B}}_{2i,2k+1} \cup \bar{\mathcal{B}}_{2i+1,2k}\right] \middle| \mathcal{B}_{i,k}\right] \\ &= \mathrm{E}\left[\bar{f}_{2k+2}\bar{f}_{2k+1}\bar{C}_{2i,2k+1} + \bar{f}_{2k+1}\bar{f}_{2k}\bar{C}_{2i+1,2k} \middle| \mathcal{B}_{i,k}\right], \end{aligned} \tag{7}$$

which leads to

$$0 = (f_k - \bar{f}_{2k+2}\bar{f}_{2k+1})\mathrm{E}\left[\bar{C}_{2i,2k+1} \middle| \mathcal{D}\right] + (f_k - \bar{f}_{2k+1}\bar{f}_{2k})\mathrm{E}\left[\bar{C}_{2i+1,2k} \middle| \mathcal{D}\right], \tag{8}$$

for any subset $\mathcal{D} \subseteq \mathcal{B}_{i,k}$.

Moreover, multiplying (7) by $(\bar{C}_{2i,2k+1} + \bar{C}_{2i+1,2k})$, resorting the terms and using (8) we get

$$0 = (f_k - \bar{f}_{2k+2}\bar{f}_{2k+1})\mathrm{Var}\left[\bar{C}_{2i,2k+1} \middle| \mathcal{D}\right] + (f_k - \bar{f}_{2k+1}\bar{f}_{2k})\mathrm{Var}\left[\bar{C}_{2i+1,2k} \middle| \mathcal{D}\right].$$

Now lets discuss under which condition the last equality together with (8) can be fulfilled:

If $(f_k - \bar{f}_{2k+2}\bar{f}_{2k+1}) = 0$ or if $(f_k - \bar{f}_{2k+1}\bar{f}_{2k}) = 0$ it follows from (8) that the corresponding other term also equals zero. This leads to $\bar{f}_{2k+2} = \bar{f}_{2k}$, which in practice usually implies $\bar{f}_{2k+2} = \bar{f}_{2k+1} = \bar{f}_{2k}$.

Otherwise, either $\mathrm{Var}\left[\bar{C}_{2i,2k+1} \middle| \mathcal{D}\right] = \mathrm{Var}\left[\bar{C}_{2i+1,2k} \middle| \mathcal{D}\right] = 0$, which means no randomness at all, or

$$\frac{\mathrm{Var}\left[\bar{C}_{2i,2k+1} \middle| \mathcal{D}\right]}{\bar{f}_{2k}\mathrm{Var}\left[\bar{C}_{2i+1,2k} \middle| \mathcal{D}\right]} = -\frac{f_k - \bar{f}_{2k+1}\bar{f}_{2k}}{\bar{f}_{2k}(f_k - \bar{f}_{2k+2}\bar{f}_{2k+1})} = \frac{\mathrm{E}\left[\bar{C}_{2i,2k+1} \middle| \mathcal{D}\right]}{\bar{f}_{2k}\mathrm{E}\left[\bar{C}_{2i+1,2k} \middle| \mathcal{D}\right]}.$$

Note, because of

$$\frac{\mathrm{E}\left[\bar{C}_{2i,2k+1}\right]}{\bar{f}_{2k}\mathrm{E}\left[\bar{C}_{2i+1,2k}\right]} = \frac{\mathrm{E}\left[\bar{C}_{2i,0}\right]}{\mathrm{E}\left[\bar{C}_{2i+1,0}\right]},$$

the fractions are not only independent of the accident year i but also of the development period k.

Therefore, except for very special cases it is not possible that the annual data $C_{i,k}$ and the half-year data $\bar{C}_{i,k}$ simultaneously fulfil the corresponding Chain-Ladder assumptions. In other words, we should not split accident years.

2.4.3 Rolling forward

If we always use the more granular data we can compare link ratios and estimated development factors, which helps in the discussion of the observed claims development result.

2.4.4 Closing

The method automatically produces estimates that can be used for the corresponding closings.

2.4.5 Forecast

The method does not give you a forecast of the next annual results.

2.4.6 Generalisation

If we want to use the method for quarterly or even monthly closings, we have to split accident years into quarters or even months, which may lead to very large triangles, see the corresponding discussion in section 2.1.

2.4.7 Usability

We have seen that the triangle may get very large and that split accident years and the annual data cannot fulfil the corresponding Chain-Ladder assumptions simultaneously. Moreover, in general we think a separate analysis of the last accident period, see section 2.7, or even a separation of half-years, see section 2.6, is much better and may be consistent with the annual analysis at year end. So why is this method still used in practice? The reasons are: On the one hand most commercial reserving software can deal easily with those huge triangles and on the other hand most actuaries do not know about the inconsistency with the annual Chain-Ladder assumptions.

2.5 Shifting of accident periods

2.5.1 Method description

This method still groups 12 accident and 12 development moth into one accident and development period, respectively. But in order to make the last observed diagonal comparable with the others we shift accident periods by six months and ignore the oldest data, see Figure 10.

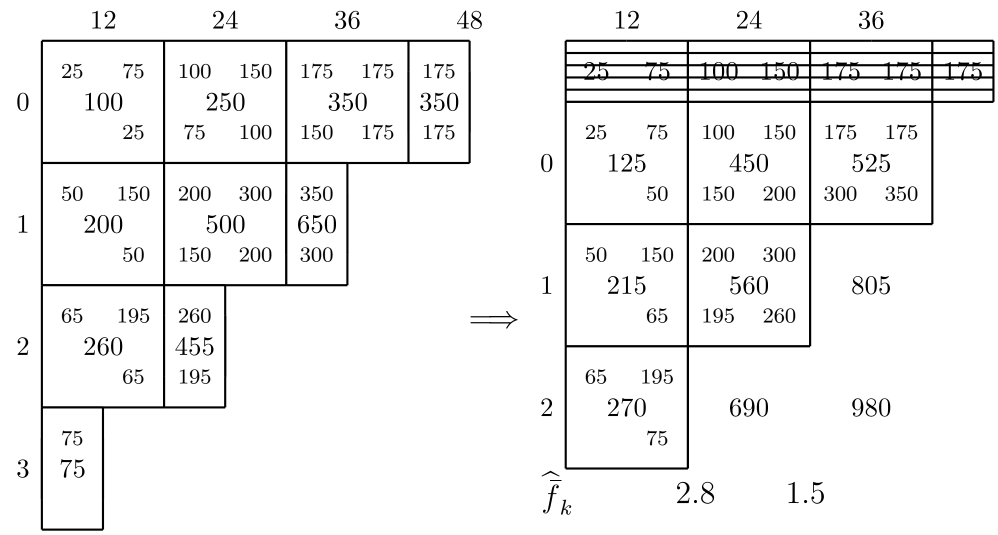

Figure 10: Shifting of accident periods.

Here we have

$$\bar{C}_{i,k} := \widetilde{C}_{2i+1,2k+1} + \widetilde{C}_{2i+2,2k}.$$

2.5.2 Theoretical consistency

Since there is no direct connection between annual data $C_{i,k}$ and shifted data $\bar{C}_{i,k}$, there might be stochastic models such that both $C_{i,k}$ and $\bar{C}_{i,k}$ fulfil the corresponding Chain-Ladder assumptions.

2.5.3 Discussion of CDR

In order to compute the claims development result with respect to the last annual closing we have to be able to estimate the ultimate claims amount for all prior accident years. Since shifting of accident periods implies that the midyear estimate $\hat{\bar{C}}_{I,I}$ of the ultimate claims amount of the last accident period contains both prior year claims and current year claims, it is not possible to calculate the claims development result. In our example we will project an ultimate claims amount of 980 for the last accident period, but the method does not lead to a split into parts that correspond to the last observed values 195 (prior year claims) and 75 (current year claims).

2.5.4 Rolling forward

For the same reasons as for the discussion of the CDR it will be very hard to use the actuarial judgement, made at the last annual closing, for the decisions at midyear. For instance, the observed link ratios are not comparable. Even the estimated development factors are only comparable in cases where we have a very stable portfolio with no material seasonal changes in the development. As we see in our very simple and deterministic example, changing volumes are enough to lead to incomparable estimated development factors.

2.5.5 Closing

The method projects an ultimate claims amount corresponding to all claims that happened up to the end of June and therefore leads to estimates which can directly be used for a midyear closing.

2.5.6 Forecast

The method cannot be used for a forecast of the annual figures at the end of June.

2.5.7 Generalisation

For an analysis at the end of another month than June we only have to shift the accident periods accordantly. Moreover, at the end of the year the shifting leads to the original annual triangle, provided the first row is not ignored.

2.5.8 Usability

Since it is not possible to calculate the claims development result with respect to the last annual closing, the method should not be used under normal circumstances. But there might be special situations, where the method is beneficially. Imagine an acquisition of a portfolio where you do not trust the booked reserves. Moreover, assume that the actuaries who might be able to explain the booked reserves are not accessible. That means irrespective of the used reserving method it is not possible to discuss the claims development result. In such situations shifting accident periods might be beneficially.

2.6 Separation of accident periods

2.6.1 Method description

Some actuaries, who use half-year or quarterly data as presented in section 2.4, argue that they can better react on seasonal effects. In our opinion, if we really believe that there are systematically seasonal effects we should separate the seasons. That leads to two triangles, $\bar{C}_{i,k}^{(1)}$ containing the accident months January to June and $\bar{C}_{i,k}^{(2)}$ containing the accident months July to December, see Figure 11.

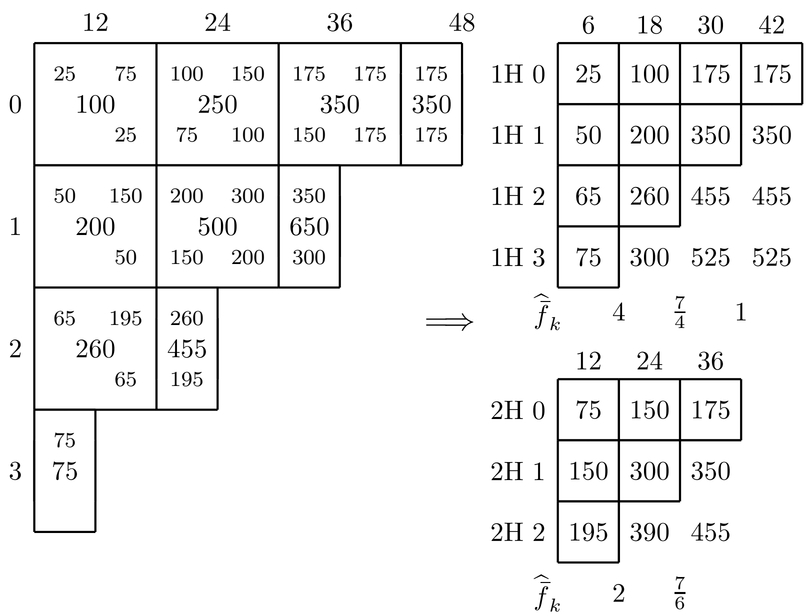

Figure 11: Separating accident periods.

In formulas we have

$$\bar{C}_{i,k}^{(1)} := \widetilde{C}_{2i,2k} \qquad \text{and} \qquad \bar{C}_{i,k}^{(2)} := \widetilde{C}_{2i+1,2k+1}.$$

2.6.2 Theoretical consistency

Since we do not make any assumption on the dependency of the two separate triangles, there is no direct connection between the annual data and the separated data. Therefore, there might be stochastic models such that the annual data and the separated data fulfil the corresponding Chain-Ladder assumptions.

2.6.3 Rolling forward

Unless we always use the separate triangles, we cannot compare link ratios or corresponding development factors with the ones of the last annual closing, which makes the discussion of the claims development very challenging.

2.6.4 Closing

The method automatically leads to estimates that can directly be used for the corresponding closings.

2.6.5 Forecast

The method does not project a forecast of the next annual result at the end of June.

2.6.6 Generalisation

In order to use this method for quarterly or even monthly projections, we have to separate quarters or even months, which leads to four or even twelve triangles and a corresponding high number of estimated development factors.

2.6.7 Usability

We think that the method is only beneficially if we do not split into too many triangles and if we use those separate triangles even for the annual closing, such that we can compare link ratios and estimated development factors in order to discuss the claims development result properly.

2.7 Separation of the last accident periods

2.7.1 Method description

The separation of seasons, described in section 2.6, is not really useful if we do get too many separate triangles. Moreover, the method of shifting accident periods and the method of extrapolation the last diagonal do not directly lead to estimates that can be used for a midyear closing. Therefore, lets try to combine the methods. For example, we could project prior years by shifting development periods, see section 2.2 and use the separation of seasons, see section 2.6, to project the triangle that contains only the accident months January to June in order to estimate the last accident period, see Figure 12.

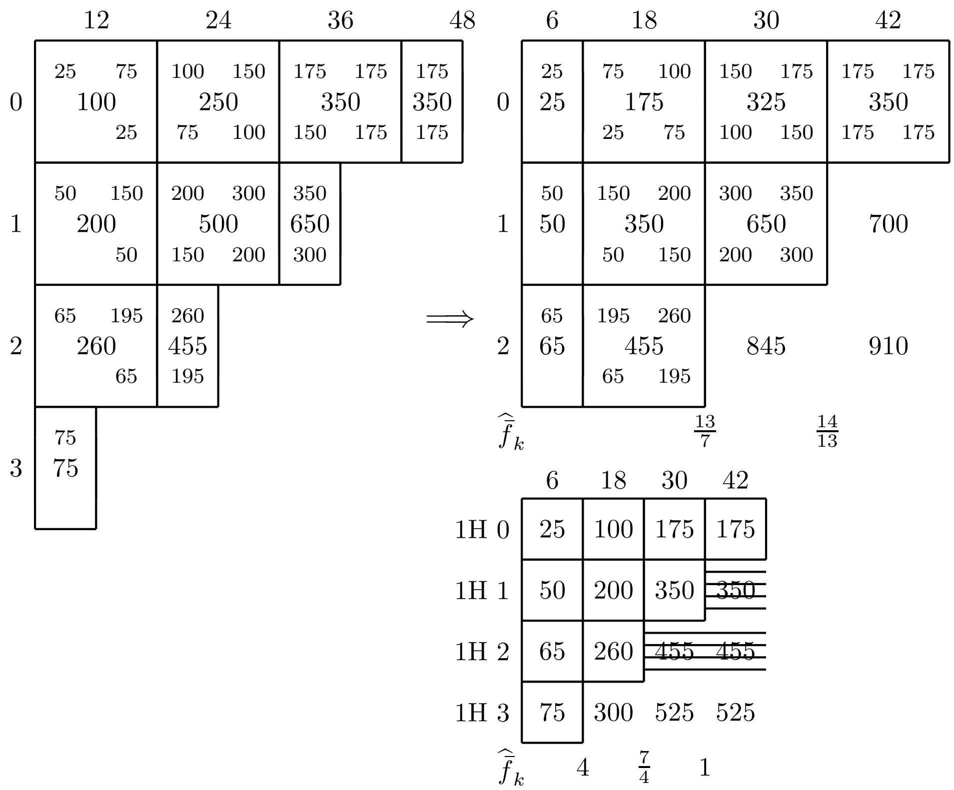

Figure 12: Splitting of accident periods.

2.7.2 Theoretical consistency

Since there is no direct connection between the annual and the more granular data their might be stochastic models such that both fulfil the corresponding Chain-Ladder assumptions.

2.7.3 Rolling forward

We have the same situation as in the case of shifted development periods, see section 2.2.

2.7.4 Closing

The estimates of this method can directly be used for the corresponding closing.

2.7.5 Forecast

The method does not lead to a forecast of the next annual closing.

2.7.6 Generalisation

The method can easily be generalised to other months than June.

2.7.7 Usability

Here we have the same situation for a midyear closing as in the case with shifted development periods for a forecast, see section 2.2.

3 Conclusion

There is no silver bullet if you want to be consistent with your reserving during a year, but under normal circumstances some methods are more suitable than others, which can be roughly summarised by

> "You may shift development or accident periods, or may split development periods, but you should not split accident periods within the same triangle."

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